ELECTRICITY AND MAGNETISM (PHYS 231)

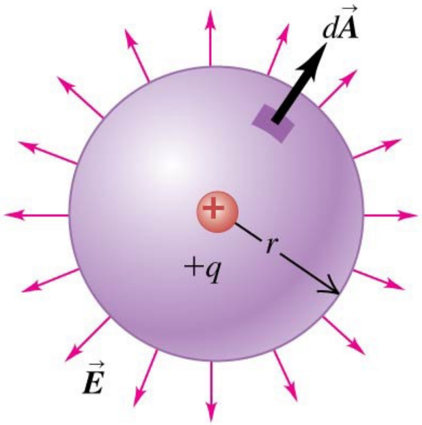
Lecture 5: Gauss's Law

Sep 4, 2024

Review on Gauss's Law

*The net flux through any **closed** surface equals the **net (total) charge** inside that surface divided by* ε_0

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$



For a point charge,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$$\oiint_S dA = 4\pi r^2$$

r can be any value.

$$\Phi_E = \frac{q}{4\pi\varepsilon_0}\oiint_S \frac{1}{r^2}\hat{r} \cdot d\vec{A}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\oiint_S dA$$

$$= \frac{q}{\varepsilon_0}$$

Work for any closed surface (sphere, cube, irregular ones...)

Review on Gauss's Law

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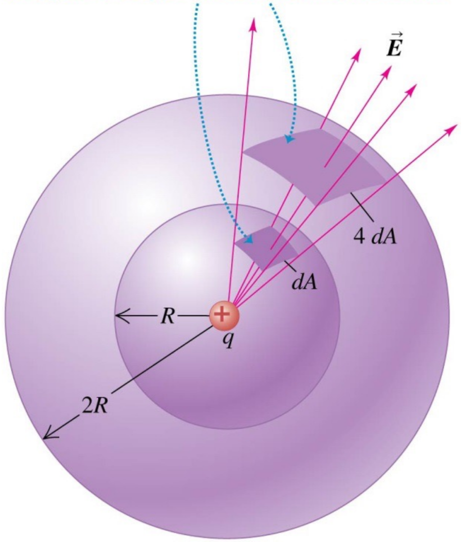

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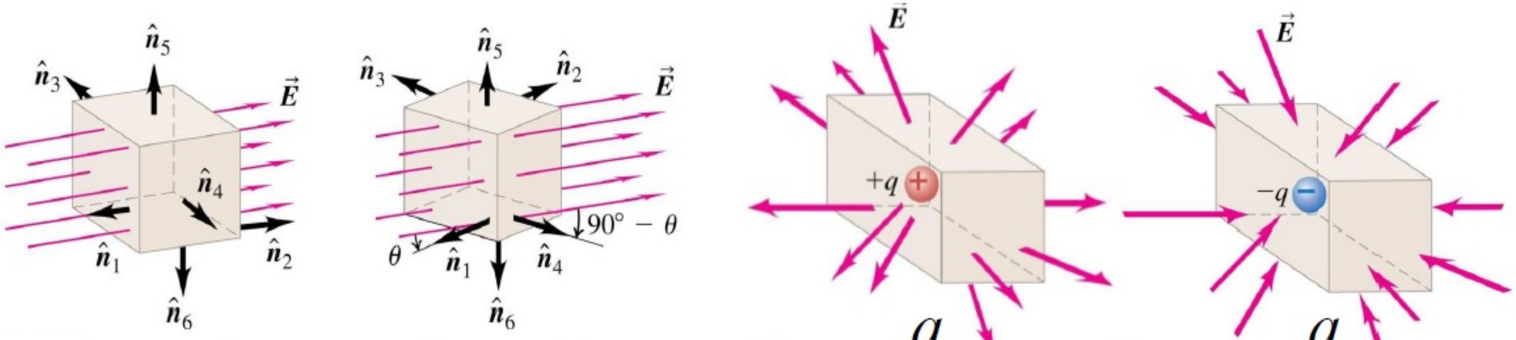


$$\Phi_E = 0 \qquad \Phi_E = 0 \qquad \Phi_E = \frac{q}{\varepsilon_0} \qquad \Phi_E = -\frac{q}{\varepsilon_0}$$

The net electric flux **ONLY** scales with the amount of charges.

Application of Gauss's Law

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

- Given an $\vec{E}$ distribution, Gauss's law can tell us the charge distribution.
- Given a charge distribution, Gauss's law can tell us the $\vec{E}$ distribution.
- Calculating $\vec{E}$ with Gauss's law **can** be much simpler than that with Coulomb's law in certain circumstances.

Coulomb or Gauss, that is THE question

Usually, but NOT always

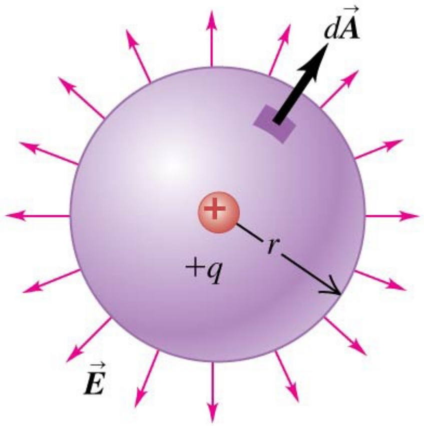
- Coulomb: Discrete Point Charge
- Gauss: Symmetric Continuous Charge Distribution

Choice of Gaussian Surface

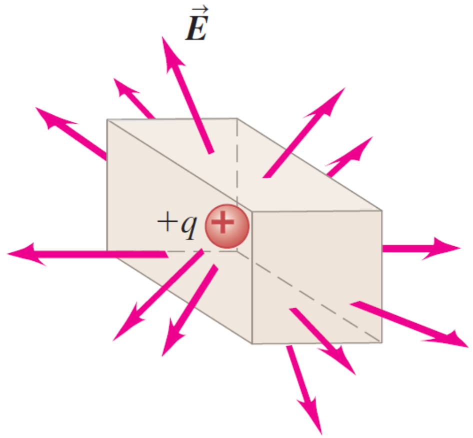
$$\Phi_E = \oint\!\!\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

Evaluation for a general surface is hard!

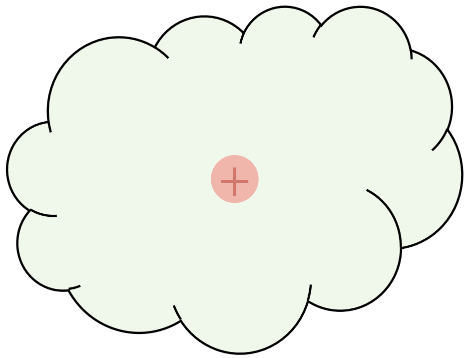
- Gaussian surface is imaginary!
- We can choose whatever we want!
- Choose the one with the **highest symmetry**!



Highest Symmetry
(**easy**)



Less Symmetry
(**hard**)



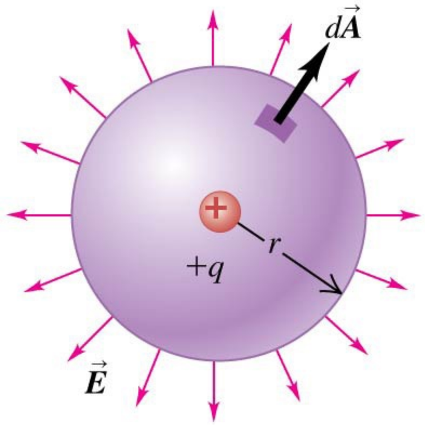
No Symmetry
(**hell**)

Example #1: Point Charge

A point charge has the highest spatial symmetry in 3D: the **spherical symmetry**!

A **spherical** Gaussian surface!

The field is always normal to the surface and pointing outward with a constant magnitude E.



$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$Q_{enclosed} = \varepsilon_0 \Phi_E = \varepsilon_0 \oiint \vec{E} \cdot d\vec{A}$$

$$= \varepsilon_0 E \oiint_S \hat{r} \cdot \hat{n} dA$$

$$= \varepsilon_0 E \oiint_S dA = 4\pi\varepsilon_0 r^2 E$$

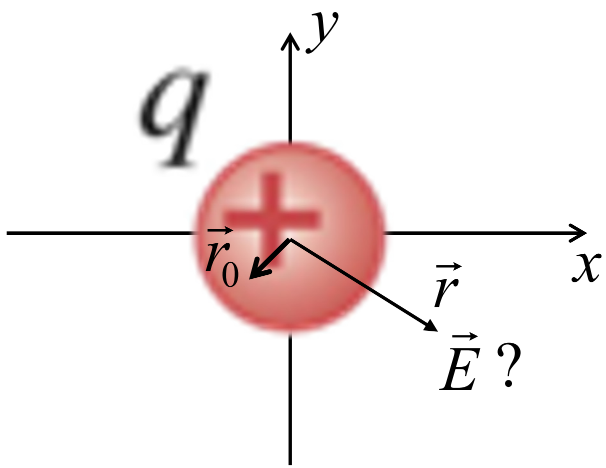
$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

This is how Coulomb's law is derived from Gauss's law.

Example #2: A Uniformly Charged Sphere

Charge q is uniformly distributed in a ball with a radius R.

1) Find $\vec{E}$ for $r > R$?
2) Find $\vec{E}$ for $r < R$?



Q: Can we use Coulomb's Law?
A: Yes, we can. But it involves complicated integral and is thus NOT recommended.

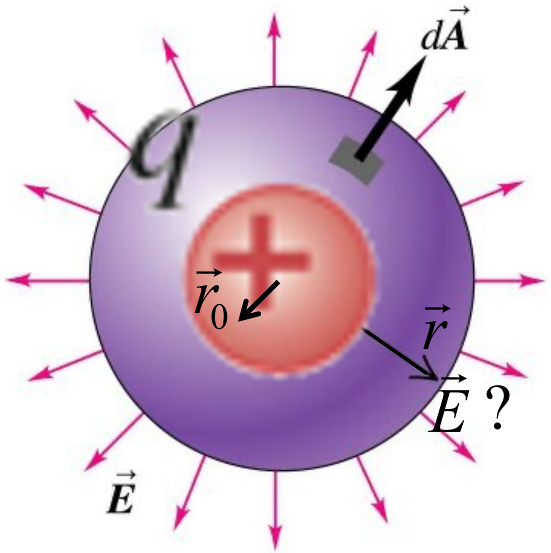
- Gauss's law is preferred.
- Make use of the **spherical symmetry**.
- Spherical Gaussian Surface for both $r > R$ & $r < R$

Example #2: $r > R$?

Charge q is uniformly distributed in a ball with a radius R.

1) Find $\vec{E}$ for $r > R$?
2) Find $\vec{E}$ for $r < R$?

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$



Step 1 Consider a spherical Gaussian surface with $r > R$.

Step 2 Note that charge enclosed is q.

Step 3 r.h.s. = q/ε_0

Step 4 Note that the field is always normal to the surface and pointing outward with a constant magnitude E.

Step 5 l.h.s. = $E\times$ Area of the Sphere = $4\pi r^2 E$

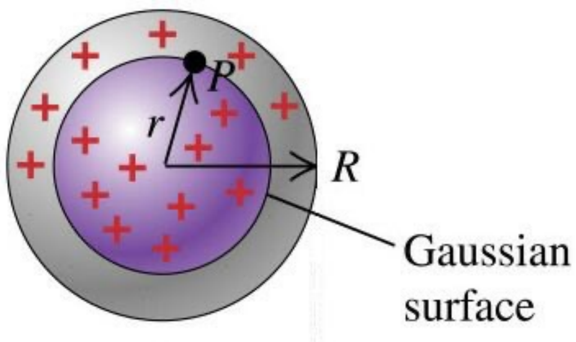
Step 6 l.h.s. = r.h.s.

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \text{ for } r > R.$$

Just like a point charge!

Example #2: $r < R$?

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$



Different from a point charge!

Step 1 Consider a spherical Gaussian surface with $r < R$.

Step 2 Calculate the charge enclosed. First, the charge density is $\rho = q/(4\pi R^3/3)$, then the charge enclosed by a radius-r sphere is

$$Q_{enclosed} = \left(\frac{4\pi r^3}{3}\right) * \rho = q\,\frac{r^3}{R^3}$$

Step 3 r.h.s. = $Q_{enclosed}/\varepsilon_0$ = $qr^3/(\varepsilon_0 R^3)$

Step 4 Note that the field is always normal to the surface and pointing outward with a constant magnitude E.

Step 5 l.h.s. = $E\times$ Area of the Sphere = $4\pi r^2 E$

Step 6 l.h.s. = r.h.s.

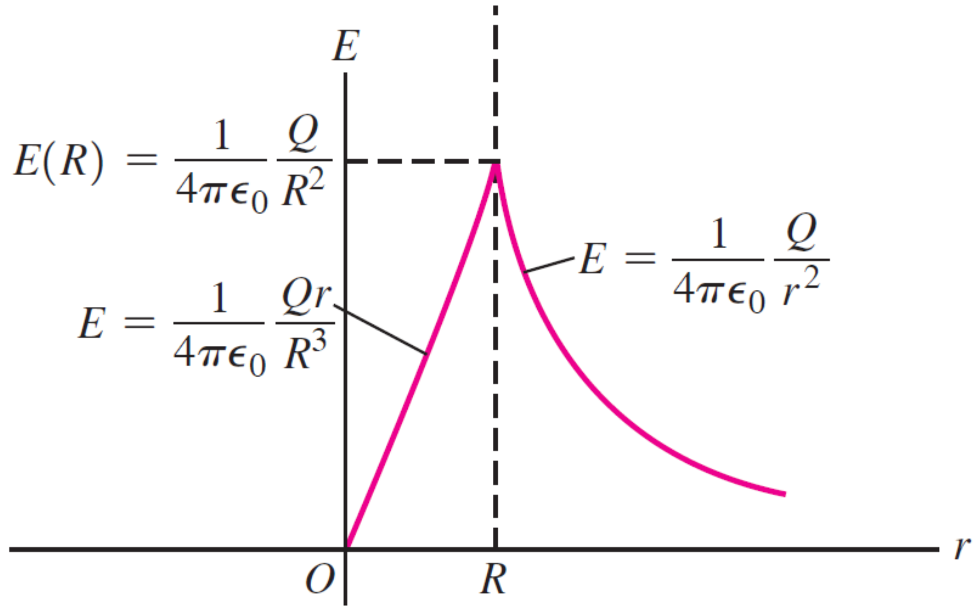
$$E = \frac{qr}{4\pi\varepsilon_0 R^3} \text{ for } r < R.$$

Example #2: Summary for a Uniformly Charged Sphere

- $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $r > R$
- $\vec{E} = \frac{qr}{4\pi\varepsilon_0 R^3}\hat{r}$ for $r < R$

The electric field inside the charged sphere linearly grows as a function of r.

The electric field outside the charged sphere decays in a power-law way (i.e. proportional to $1/r^2$).



Recipe For Calculating $\vec{E}$ with Gauss's Law

Step 1 Identify the Symmetry of the charged system.

Step 2 Carefully choose a Gaussian surface to exploit the symmetries.

Step 3 Identify the total charge enclosed $Q_{enclosed}$.

Step 4 r.h.s. = $Q_{enclosed}/\varepsilon_0$

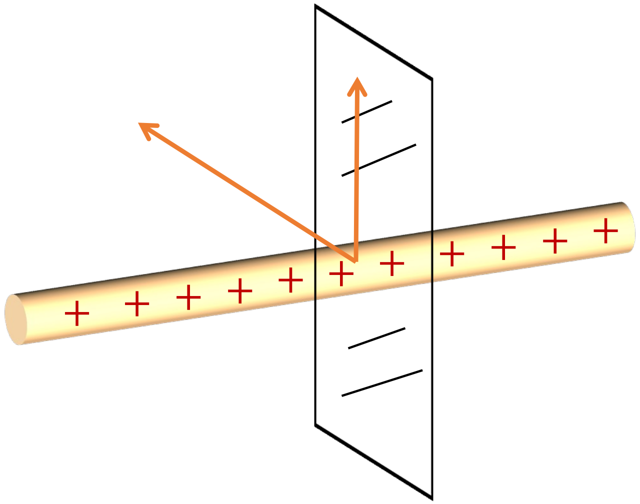
Step 5 Calculate the electric flux, i.e. the l.h.s. of the equation. A clever choice of the Gaussian surface can greatly simplify the flux integral.

Step 6 l.h.s. = r.h.s.

Step 7 Solve for magnitude of $\vec{\boldsymbol{E}}$ and its direction.

Example #3: An Infinitely Long Wire

Q: Electric charge is distributed uniformly along an infinitely long, thin wire. The charge per unit length is λ. Find the electric field by using Gauss's law.

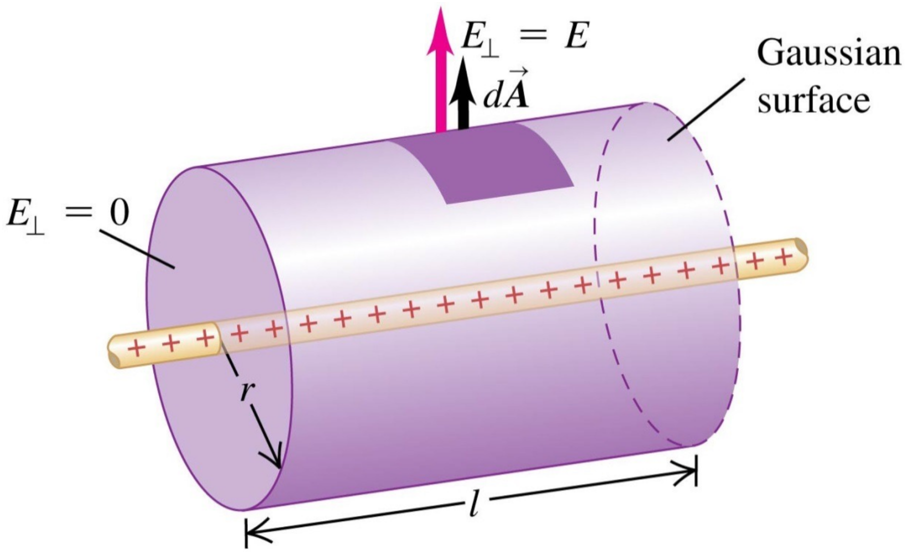


- A wire has **cylindrical symmetry**.
- The electric field of an infinitely long + uniformly charged wire can **ONLY point radially** outward for $\lambda > 0$ (or inward for $\lambda < 0$). **Why?**

- An infinitely long wire additionally has a **mirror reflection** symmetry!
- The mirror is a plane normal to the wire.
- Only radially outward/inward electric field is **compatible with the reflection symmetry**.

Example #3: An Infinitely Long Wire

Q: Electric charge is distributed uniformly along an infinitely long, thin wire. The charge per unit length is $\lambda > 0$. Find the electric field by using Gauss's law.

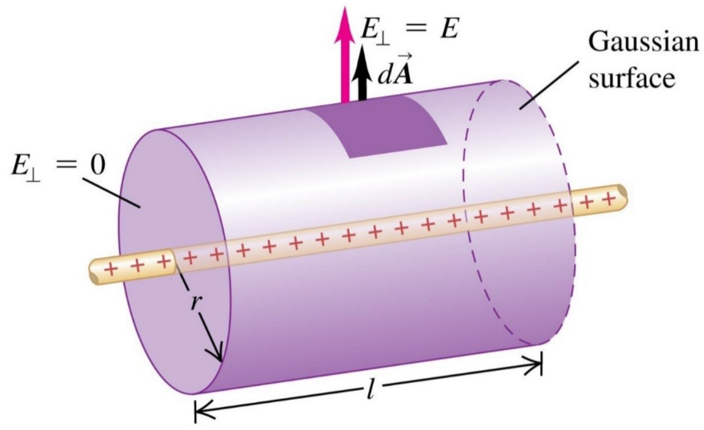


- A wire has cylindrical symmetry.
- The electric field of an infinitely long + uniformly charged wire can ONLY point radially outward for $\lambda > 0$ (or inward for $\lambda < 0$).
- Consider a cylindrical Gaussian surface of radius r and length l.
- $\vec{E}$ is normal to the side surface of the cylinder and maintain a constant magnitude E

Example #3: An Infinitely Long Wire

Q: Electric charge is distributed uniformly along an infinitely long, thin wire. The charge per unit length is $\lambda > 0$. Find the electric field by using Gauss's law.

- Consider a **cylindrical** Gaussian surface of radius r and length l.
- $\vec{E}$ is **normal** to the Gaussian surface and maintain a **constant** magnitude E



$$\Phi_E = \oint\!\!\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

r.h.s. = $\frac{Q_{enclosed}}{\varepsilon_0} = \lambda l / \varepsilon_0$

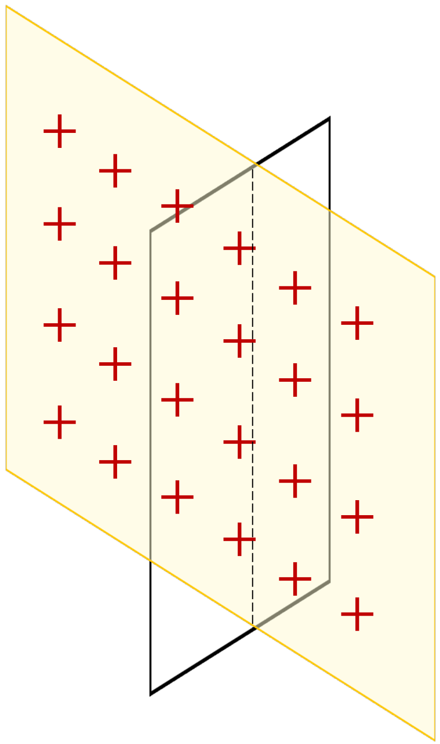
l.h.s. = $E \times$ **side surface area** of the cylinder = $\mathbf{2\pi r l}\, E$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\,\hat{r}$$

Example #4: An Infinite Plane Sheet of Charge

Q: Use Gauss's law to find the electric field caused by a thin, flat, infinite sheet with a uniform positive surface charge density $\sigma > 0$.

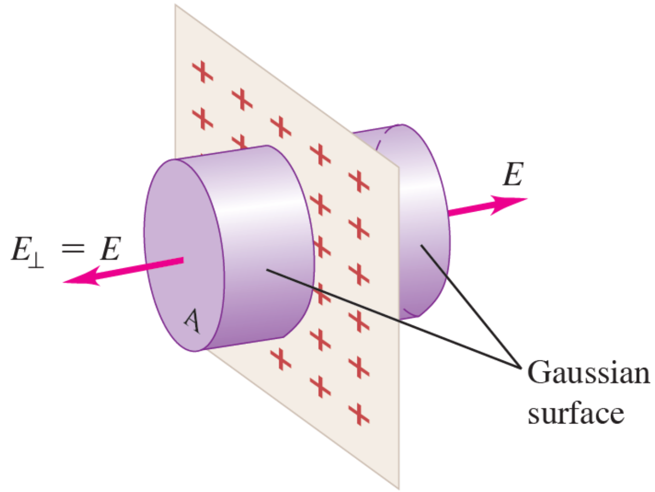
Try to convince yourself that the electric field direction **MUST** be perpendicular to the charged plane, because there exists an infinite number of **mirror reflection symmetries** (like the one shown here) for an infinitely large plane.



Example #4: An Infinitely Plane Sheet of Charge

Q: Use Gauss's law to find the electric field caused by a thin, flat, infinite sheet with a uniform positive surface charge density $\sigma > 0$.

- Consider a **cylindrical** Gaussian surface of radius r and length l.
- $\vec{E}$ is **normal** to the top & bottom surface of the cylinder and maintain a **constant** magnitude E



$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

r.h.s. = $\frac{Q_{enclosed}}{\varepsilon_0} = \pi r^2 \sigma / \varepsilon_0$

l.h.s. = $E \times$ **top & bottom surface areas** of the cylinder

= $2 \times \pi r^2 \, E$

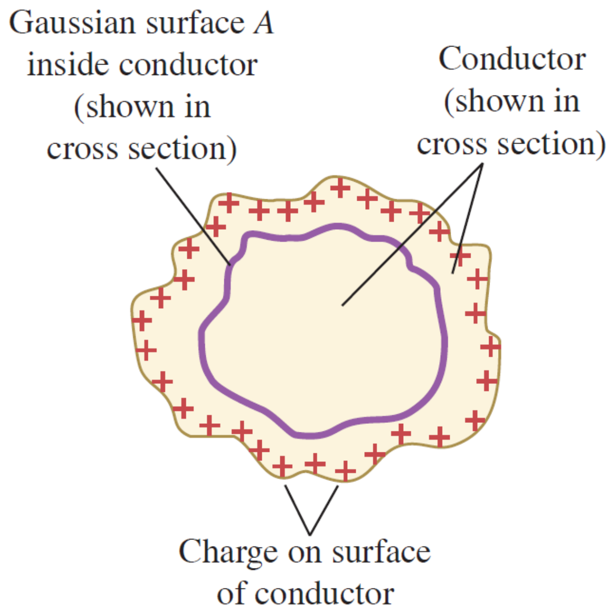
$$E = \frac{\sigma}{2\varepsilon_0}$$

Charge on Conductors

Fact #1: In the *electrostatic* limit (no movement of charges), the electric field inside a conductor will be everywhere zero.

(easy to understand: no movement --> no force --> no field)

Fact #2: When *excess charge* is placed on a solid conductor and is at rest, it resides entirely *on the surface*, NOT in the interior of the material.



1) The zero-field-condition (fact #1) indicates an arbitrary Gaussian surface inside the conductor will necessarily have zero flux.
2) Namely, an arbitrary Gaussian surface will enclose zero net charge.
3) No excess charge can live in the interior of the conductor.
4) Excess charge can **ONLY** live on the surface.

Field at the Surface of a Conductor

Electric field at surface of a conductor, $\vec{E}$ perpendicular to surface → $E_\perp = \dfrac{\sigma}{\epsilon_0}$ ← σ: Surface charge density; ϵ_0: Electric constant

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

Consider a cylindrical Gaussian surface with a radius r

l.h.s. = $\Phi_E = E\,\pi r^2$ (only top surface contributes)

r.h.s. = $Q_{enclosed}/\varepsilon_0 = \pi r^2 \sigma/\varepsilon_0$