Department of Mathematics – University of Tennessee

Math 251 Matrix Algebra I

Test 3

Name: Solutions

Time allowed: 50 minutes

Instructions:

- Calculators are not allowed.
- All electronic devices must be put away.
- Answers with insufficient or incorrect working will not receive full credit.
- Simplify answers whenever possible.

Page	Points	Score
2	10	
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6	10	
Total:	50	

1. (10 pts) Decide whether each statement is TRUE or FALSE. *No justification is required.*

- The set of all polynomials with the usual addition and scalar multiplication is a vector space.

  True.

- The intersection of two subspaces of a vector space V is also a subspace of V.

  True.

- If S is a set of vectors in a vector space V, then $\text{span}(S)$ is a subspace of V.

  True.

- The span of a nonzero vector in $\mathbb{R}^3$ is a line.

  True.

- The polynomials $x-1$, $(x-1)^2$, and $(x-1)^3$ span P_3.

  False.

- If W is a set of vectors containing the zero vector, then W is linearly independent.

  False.

- Every set of n linearly independent vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

  True.

- If B is the standard basis for $\mathbb{R}^n$ and $\vec{v}$ is in $\mathbb{R}^n$, then $[\vec{v}]_B = \vec{v}$.

  True.

- If W is a subspace of a finite-dimensional vector space V, then $\dim W \leq \dim V$.

  True.

- If A is a 3×3 matrix, then $A = P_{B_1 \to B_2}$ for some bases B_1 and B_2 of $\mathbb{R}^3$.

  False.

2. (6 pts) Let V be the set of all pairs $\vec{v} = (a, b)$ of real numbers with the following operations:

$$(a, b) + (c, d) = (a + c, 0),$$
$$k(a, b) = (ka, kb).$$

(a) Does V satisfy axiom 2 for vector spaces? Prove or give a counterexample.

Yes.

$(a,b) + (c,d) = (a+c, 0) = (c+a, 0) = (c,d) + (a,b).$

(b) Does V satisfy axiom 8 for vector spaces? Prove or give a counterexample.

No.

$(1+1)(1,1) = 2(1,1) = (2,2)$

$1(1,1) + 1(1,1) = (1,1) + (1,1) = (2,0).$

Thus $(1+1)(1,1) \neq 1(1,1) + 1(1,1).$

3. (4 pts) Determine whether the set of polynomials of the form $p(x) = a + bx^2$ is a subspace of P_2.

$(a + bx^2) + (c + dx^2) = (a+c) + (b+d)x^2,$

so the set is closed under addition.

$k(a + bx^2) = (ka) + (kb)x^2,$

so the set is closed under scalar multiplication.

Thus the set of polynomials of the form $a + bx^2$ is a subspace of P_2.

4. (10 pts) Consider the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Determine whether A, B, and C span M_{22}.

$$k_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{cases} k_1 + k_2 = a \\ k_1 + k_3 = b \\ k_2 + k_3 = c \\ k_1 + k_2 = d \end{cases} \Rightarrow a = d.$$

Thus A, B, C do <u>not</u> span M_{22}.

(b) Determine whether A, B, and C are linearly independent in M_{22}.

from part (a):

$$\begin{cases} k_1 + k_2 = 0 \\ k_1 + k_3 = 0 \\ k_2 + k_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\Rightarrow k_1 = 0, \ k_2 = 0, \ k_3 = 0.$$

Thus A, B, C are linearly independent.

(c) Is the set $\{A, B, C\}$ a basis for M_{22}? Explain briefly.

<u>No</u>. The set $\{A, B, C\}$ does <u>not</u> span M_{22}, so is not a basis for M_{22}.

5. (10 pts) Let $B_1 = \{(1,1),(1,-1)\}$ and $B_2 = \{(0,1),(1,2)\}$ be two bases for $\mathbb{R}^2$.

(a) Find the coordinate vector $[\vec{v}]_{B_1}$ with respect to B_1 for $\vec{v} = (2,4)$.

$$\vec{v} = (2,4) = 3(1,1) + (-1)(1,-1),$$

so $$[\vec{v}]_{B_1} = (3,-1).$$

(b) Find the transition matrix $P_{B_1 \to B_2}$.

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$P_{B_1 \to B_2} = \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}.$$

(c) Use your answer from part (a) to find $[\vec{v}]_{B_2}$.

$$[\vec{v}]_{B_2} = P_{B_1 \to B_2}[\vec{v}]_{B_1} = \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

(d) Does your answer to part (c) match what you expect? Explain briefly.

<u>Yes</u>. $\vec{v} = 0(0,1) + 2(1,2)$, so

$$[\vec{v}]_{B_2} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$ by definition.

6. (6 pts) Consider the linear system below.

$$\begin{aligned} x_1 - 2x_2 + 2x_3 \phantom{{}- x_4} &= 0 \\ x_1 - x_2 + 2x_3 - x_4 &= 0 \\ x_2 \phantom{{}+2x_3} - x_4 &= 0 \end{aligned}$$

The general solution for this system is $x_1 = -2s + 2t$, $x_2 = t$, $x_3 = s$, $x_4 = t$.

(a) Find a basis for the solution space of the linear system.

The general solution is

$$\vec{x} = (-2s + 2t, t, s, t)$$
$$= s(-2, 0, 1, 0) + t(2, 1, 0, 1).$$

A basis for the solution space is

$$B = \{(-2, 0, 1, 0), (2, 1, 0, 1)\}.$$

(b) What is the dimension of this solution space? Explain briefly.

The dimension is 2, because the basis in part (a) contains two vectors.

7. (4 pts) State the dimension of each vector space. *No justification is required.*

(a) $\mathbb{R}^5$

5

(b) P_3

4

(c) The set of all 3×3 lower triangular matrices.

6

(d) The subspace of $\mathbb{R}^3$ spanned by the vectors $\vec{u} = (1, -1, 0)$ and $\vec{v} = (-1, 1, 0)$.

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