Department of Mathematics – University of Tennessee

Math 251 Matrix Algebra I

Test 3 Practice

Name: Solutions

Time allowed: 50 minutes

Instructions:

- Calculators are not allowed.
- All electronic devices must be put away.
- Answers with insufficient or incorrect working will not receive full credit.
- Simplify answers whenever possible.

Page	Points	Score
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	50	

1. (10 pts) Decide whether each statement is TRUE or FALSE. *No justification is required.*

- $\mathbb{R}^n$ with the usual addition and scalar multiplication is a vector space.

True.

- A vector space must contain at least two vectors.

False.

- The set of all invertible $n \times n$ matrices is a subspace of M_{nn}.

False.

- The union of two subspaces of a vector space V is also a subspace of V.

False.

- The span of a finite set of vectors in a vector space V is closed under addition.

True.

- Every linearly independent set contains the zero vector.

False.

- If $V = \text{span}(S)$, then S is a basis for V.

False.

- Every linearly independent subset of a vector space V is a basis for V.

False.

- Every set of four vectors that span $\mathbb{R}^4$ is a basis for $\mathbb{R}^4$.

True.

- Transition matrices are invertible.

True.

2. (6 pts) Let V be the set of all pairs $\vec{v} = (a, b)$ of real numbers with the following operations:

$$(a,b) + (c,d) = (a+c, b-d),$$
$$k(a,b) = (ka, 0).$$

(a) Does V satisfy axiom 2 for vector spaces? Prove or give a counterexample.

No. $(0,1) + (1,0) = (1,1)$ and $(1,0) + (0,1) = (1,-1)$,

so $(0,1) + (1,0) \neq (1,0) + (0,1)$.

(b) Does V satisfy axiom 8 for vector spaces? Prove or give a counterexample.

Yes.
$$\begin{aligned}(k+m)(a,b) &= ((k+m)a, 0)\\ &= (ka + ma, 0)\\ &= (ka, 0) + (ma, 0)\\ &= k(a,b) + m(a,b).\end{aligned}$$

3. (4 pts) Determine whether the set of 2×2 matrices $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ such that $x + z = 0$ is a subspace of M_{22}.

If $x_1 + z_1 = 0$ and $x_2 + z_2 = 0$, then

$$\begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix} + \begin{bmatrix} x_2 & y_2 \\ 0 & z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ 0 & z_1 + z_2 \end{bmatrix}$$

where $(x_1 + x_2) + (z_1 + z_2) = (x_1 + z_1) + (x_2 + z_2) = 0 + 0 = 0$.

If $x + z = 0$, then

$$k \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} kx & ky \\ 0 & kz \end{bmatrix}$$

where $kx + kz = k(x+z) = k(0) = 0$.

This set is closed under addition and scalar multiplication, so it is a subspace of M_{22}.

4. (10 pts) Let $S = \{1, 1+x, 1-x+x^2\}$.

Recall that P_2 is the set of all polynomials with degree at most 2.

(a) Determine whether S spans P_2.

$$k_1(1) + k_2(1+x) + k_3(1-x+x^2) = a_0 + a_1 x + a_2 x^2$$

$$\Rightarrow \begin{cases} k_1 + k_2 + k_3 = a_0 \\ k_2 - k_3 = a_1 \\ k_3 = a_2 \end{cases} \Rightarrow \begin{cases} k_1 = a_0 - a_1 - 2a_2 \\ k_2 = a_1 + a_2 \\ k_3 = a_2 \end{cases}$$

This system is consistent for all choices of a_0, a_1, a_2.

Thus S spans P_2.

(b) Determine whether S is linearly independent in P_2.

From part (a), the only solution to

$$k_1(1) + k_2(1+x) + k_3(1-x+x^2) = 0$$

is the trivial solution $k_1 = 0$, $k_2 = 0$, $k_3 = 0$.

Thus S is linearly independent.

(c) Is S a basis for P_2? Explain briefly.

<u>Yes.</u> S is linearly independent and spans P_2, so S is a basis for P_2.

5. (10 pts) Let $B_1 = \{(1,0),(1,1)\}$ and $B_2 = \{(0,1),(1,-1)\}$ be two bases for $\mathbb{R}^2$.

(a) Find the coordinate vector $[\vec{v}]_{B_1}$ with respect to B_1 for $\vec{v} = (3,1)$.

$$\vec{v} = (3,1) = 2(1,0) + 1(1,1),$$

$$\text{so} \quad [\vec{v}]_{B_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(b) Find the transition matrix $P_{B_1 \to B_2}$.

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$P_{B_1 \to B_2} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

(c) Use your answer from part (a) to find $[\vec{v}]_{B_2}$.

$$[\vec{v}]_{B_2} = P_{B_1 \to B_2} [\vec{v}]_{B_1}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

6. (10 pts) Describe the solution space for each linear system as a subspace of $\mathbb{R}^3$, and state the dimension of each solution space.

(a) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is the plane through the origin with normal $(1, -2, 1)$.

i.e. all points $x = 2s - t$, $y = s$, $z = t$.

The dimension is 2.

(b) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is the line through the origin parallel to $(3, -2, -1)$.

i.e. all points $x = 3t$, $y = -2t$, $z = -t$.

The dimension is 1.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is the point $(0, 0, 0)$.

The dimension is 0.