Department of Mathematics – University of Tennessee

Math 251 Matrix Algebra I

Test 2

Name: Solutions

Time allowed: 50 minutes

Instructions:

- Calculators are not allowed.
- All electronic devices must be put away.
- Answers with insufficient or incorrect working will not receive full credit.
- Simplify answers whenever possible.

Page	Points	Score
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	50	

1. (10 pts) Decide whether each statement is TRUE or FALSE. *No justification is required.*

- If A and B are square matrices of the same size, then $\det(A+B) = \det A + \det B$.

  False.

- If A and B are square matrices of the same size, then $\det(AB) = (\det A)(\det B)$.

  True.

- If A is an invertible matrix, then $\det A \neq 0$.

  True.

- The adjoint of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\operatorname{adj} A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$.

  False.

- If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

  True.

- If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$.

  True.

- If $\vec{a}$ and $\vec{b}$ are orthogonal, then $\operatorname{proj}_{\vec{a}}(\operatorname{proj}_{\vec{b}}(\vec{u})) = \vec{0}$.

  True.

- If $\vec{x}$ is a solution to the linear system $A\vec{x} = \vec{b}$, then $\vec{x}$ is orthogonal to every row of the matrix A.

  False.

- The equation of a plane can be determined from any three distinct points on the plane.

  False.

- If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$, then $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \, \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$.

  True.

2. (5 pts) Find the determinant of the matrix A using cofactor expansion down the first column.

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 5 & 1 & -3 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 3 & -2 & 0 \\ 5 & 1 & -3 \\ 0 & -1 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} - 5\begin{vmatrix} -2 & 0 \\ -1 & 1 \end{vmatrix} + 0\begin{vmatrix} -2 & 0 \\ 1 & -3 \end{vmatrix}$$

$$= 3(1-3) - 5(-2-0) + 0$$

$$= 3(-2) - 5(-2)$$

$$= \underline{4.}$$

3. (5 pts) Find the determinant of the matrix B.

$$B = \begin{bmatrix} 0 & -4 & 4 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 2 & 0 & -1 \\ 3 & 0 & -2 & 2 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 0 & -4 & 4 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 2 & 0 & -1 \\ 3 & 0 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 4 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 2 & 0 & -1 \\ 3 & 0 & -2 & 2 \end{vmatrix} = 4\begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= 4\begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 4(3)(2)(1) = \underline{24.}$$

4. (4 pts) Suppose that A is a 3×3 matrix with $\det A = 4$.

(a) Find $\det(2A)$.

$$\det 2A = 2^3 \det A = 8 \cdot 4 = \underline{32}.$$

(b) Find $\det B$, where B is obtained from A by replacing row 1 with the sum of row 1 and row 2.

$$\det B = \det A = \underline{4}.$$

(c) Find $\det C$, where C is obtained from A by swapping row 1 and row 2.

$$\det C = -\det A = \underline{-4}.$$

(d) Find $\det(A^T)$.

$$\det A^T = \det A = \underline{4}.$$

5. (6 pts) Let $\vec{u} = (2, 0, 1)$ and $\vec{v} = (-1, 2, 1)$. Compute each of the following, or explain why it is not possible.

(a) $(\vec{u} \cdot \vec{v})\vec{v} = \big((2,0,1)\cdot(-1,2,1)\big)(-1,2,1)$

$= (-1)(-1,2,1)$

$= \underline{(1,-2,-1)}.$

(b) $\|\vec{u} + \vec{v}\| = \|(2,0,1) + (-1,2,1)\|$

$= \|(1,2,2)\|$

$= \sqrt{1^2 + 2^2 + 2^2} = \underline{3}.$

(c) $\vec{u} \times (\vec{u} \cdot \vec{v}) =$ not possible.

$\vec{u}$ is a vector, and $\vec{u}\cdot\vec{v}$ is a scalar.

You cannot take the cross product of a vector and a scalar.

6. (5 pts) Consider the two vectors $\vec{a} = (1, -1, -1, 1)$ and $\vec{b} = (2, -2, 0, -1)$. Find the component of $\vec{b}$ parallel to $\vec{a}$ and the component of $\vec{b}$ orthogonal to $\vec{a}$.

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} = \frac{(2,-2,0,-1)\cdot(1,-1,-1,1)}{\|(1,-1,-1,1)\|^2}(1,-1,-1,1)$$

$$= \frac{2+2+0-1}{(\sqrt{4})^2}(1,-1,-1,1)$$

$$= \frac{3}{4}(1,-1,-1,1) = \left(\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{3}{4}\right).$$

$$\vec{b} - \text{proj}_{\vec{a}} \vec{b} = (2,-2,0,-1) - \left(\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{3}{4}\right) = \left(\frac{5}{4}, -\frac{5}{4}, \frac{3}{4}, -\frac{7}{4}\right).$$

The component parallel to $\vec{a}$ is $\underline{\left(\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{3}{4}\right)}$.

The component orthogonal to $\vec{a}$ is $\underline{\left(\frac{5}{4}, -\frac{5}{4}, \frac{3}{4}, -\frac{7}{4}\right)}$.

7. (5 pts) Conisder the vector $\vec{v} = (3, -2, -2)$ in $\mathbb{R}^3$ and the point $\vec{x}_0 = (3, 6, -1)$.

(a) Find a scalar equation for the plane containing $\vec{x}_0$ and orthogonal to $\vec{v}$.

$$\vec{v}\cdot(\vec{x} - \vec{x}_0) = 0 \Rightarrow (3,-2,-2)\cdot(x-3, y-6, z+1) = 0$$

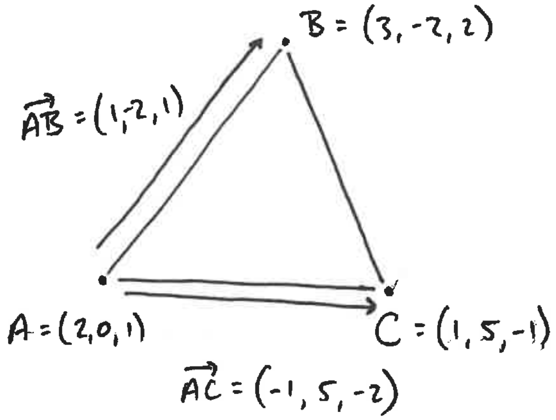
$$\Rightarrow 3(x-3) - 2(y-6) - 2(z+1) = 0$$

$$\Rightarrow \underline{3x - 2y - 2z = -1}.$$

(b) Find a vector equation for the line through $\vec{x}_0$ and parallel to $\vec{v}$.

$$\vec{x} = \vec{x}_0 + t\vec{v} = \underline{(3,6,-1) + t(3,-2,-2)}.$$

8. (5 pts) Find the area of the triangle in $\mathbb{R}^3$ with vertices $A = (2, 0, 1)$, $B = (3, -2, 2)$, and $C = (1, 5, -1)$.



$$\text{area} = \frac{1}{2} \left\| \vec{AB} \times \vec{AC} \right\|$$

$$\vec{AB} \times \vec{AC} = (1, -2, 1) \times (-1, 5, -2) = (-1, 1, 3)$$

$$\text{area} = \frac{1}{2} \left\| (-1, 1, 3) \right\| = \underline{\frac{1}{2}\sqrt{11}}.$$

9. (5 pts) Suppose that $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ with $\|\vec{u}\| = \|\vec{v}\|$. Show that $\vec{u} + \vec{v}$ is orthogonal to $\vec{u} - \vec{v}$.

$$\begin{aligned}
(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) &= \vec{u} \cdot (\vec{u} - \vec{v}) + \vec{v} \cdot (\vec{u} - \vec{v}) \\
&= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} \\
&= \|\vec{u}\|^2 - \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{v} - \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 - \|\vec{u}\|^2 \\
&= 0.
\end{aligned}$$

$\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are orthogonal because $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$.