Department of Mathematics – University of Tennessee

Math 251 Matrix Algebra I

Test 2 Practice

Name: Solutions

Time allowed: 50 minutes

Instructions:

- Calculators are not allowed.
- All electronic devices must be put away.
- Answers with insufficient or incorrect working will not receive full credit.
- Simplify answers whenever possible.

Page	Points	Score
2	10	
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6	10	
Total:	50	

1. (10 pts) Decide whether each statement is TRUE or FALSE. *No justification is required.*

- The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

True.

- The determinant of an upper triangular matrix is the sum of the entries on the main diagonal.

False. [This should be the product of the diagonal entries.]

- If $\det A = 5$ then $\det(2A) = 10$.

False, [If A is an $n \times n$ matrix, then $\det(kA) = k^n \det A$.]

- If A is an $n \times n$ matrix, then $A(\mathrm{adj} A) = (\det A)\, I_n$.

True.

- If $\vec{u} + \vec{v} = \vec{u} + \vec{w}$, then $\vec{v} = \vec{w}$.

True.

- If $\vec{u} \cdot \vec{v} = 0$, then either $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

False. [If $\vec{u} = (1,0)$ and $\vec{v} = (0,1)$, then $\vec{u} \cdot \vec{v} = 0$.]

- If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then $\|\vec{u} + \vec{v}\| \geq \|\vec{u}\| + \|\vec{v}\|$.

False. [The triangle inequality says $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.]

- The vectors $(1, 2, 3)$ and $(-3, 2, -1)$ are orthogonal.

False. [$(1,2,3) \cdot (-3,2,-1) = -3 + 4 - 3 = -2 \neq 0$.]

- A vector equation for a line can be determined from any point on the line and any nonzero vector parallel to the line.

True.

- If $\vec{u}$ is any vector in $\mathbb{R}^3$, then $\vec{u} \times \vec{u} = \vec{0}$.

True.

2. (5 pts) Find the determinant of the matrix A.

$$A = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 3 & 2 & 1 \\ 1 & 3 & -1 & 0 \\ 2 & -2 & 1 & 0 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 3 & 3 & 1 \\ 0 & 3 & 2 & 1 \\ 1 & 3 & -1 & 0 \\ 2 & -2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 1 & 3 & -1 & 0 \\ 2 & -2 & 1 & 0 \end{vmatrix}$$

expand along column 4

$$= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 3 & -1 \\ 2 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix}$$

$$= (3-(+2)) + (-2-6) = \underline{-\cancel{4}7}$$

3. (5 pts) Use Cramer's Rule to find the value of y given the linear system below.

$$\begin{aligned} x + 3y - z &= 2 \\ 2x - 4y + z &= 0 \\ 2x + y - 8z &= 5 \end{aligned}$$

$$\det A = \begin{vmatrix} 1 & 3 & -1 \\ 2 & -4 & 1 \\ 2 & 1 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -10 & 3 \\ 0 & -5 & -6 \end{vmatrix} = 60 + 15 = 75$$

$$\det A_y = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 2 & 5 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 1 & -6 \end{vmatrix} = 24 - 3 = 21$$

$$y = \frac{\det A_y}{\det A} = \frac{21}{75} = \underline{\frac{7}{25}}.$$

4. (10 pts) Let $\vec{u} = (2, -2, 1)$, $\vec{v} = (3, 3, 1)$, and $\vec{w} = (0, 4, -2)$. Compute each of the following, or explain why it is not possible.

(a) $\vec{u} \cdot \vec{v} =$ $(2,-2,1)\cdot(3,3,1) = 6-6+1 = \underline{1}$.

(b) $\|\vec{u}\| =$ $\sqrt{(2)^2+(-2)^2+(1)^2} = \sqrt{9} = \underline{3}$.

(c) $\text{proj}_{\vec{u}}\vec{v} =$ $\dfrac{\vec{u}\cdot\vec{v}}{\|u\|^2}\vec{u} = \frac{1}{9}(2,-2,1) = \underline{\left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right)}$.

(d) $\vec{u} + (\vec{v} \cdot \vec{w}) =$ not possible.

The first term is a vector and the second term is a scalar. You cannot add a vector and a scalar.

(e) $\vec{u} + (\vec{v} \times \vec{w}) =$ $(2,-2,1) + ((3,3,1)\times(0,4,-2))$

$= (2,-2,1) + (-10,6,12)$

$= \underline{(-8,4,13)}$.

$$\left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & 1 \\ 0 & 4 & -2 \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ 3 & 3 \\ 0 & 4 \end{matrix}$$

$= -6\vec{i} + 12\vec{k} - 4\vec{i} + 6\vec{j}$

$= (-10, 6, 12)$

5. (5 pts) Find the component of $\vec{u} = (1,2,3,4)$ that is parallel to $\vec{a} = (1,1,1,1)$ and the component of $\vec{u}$ that is orthogonal to $\vec{a}$.

$\vec{u} \cdot \vec{a} = (1,2,3,4) \cdot (1,1,1,1) = 10$, $\|\vec{a}\| = \sqrt{4} = 2$

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} = \frac{10}{4}(1,1,1,1) = \left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right).$$

$$\vec{u} - \text{proj}_{\vec{a}} \vec{u} = (1,2,3,4) - \left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right) = \left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right).$$

The component parallel to $\vec{a}$ is $\left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right)$.

The component orthogonal to $\vec{a}$ is $\left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$.

6. (5 pts) Conisder the plane in $\mathbb{R}^3$ that is parallel to the vector $\vec{v} = (1,1,1)$ and contains both of the points $\vec{x}_0 = (2,6,-1)$ and $\vec{x}_1 = (-1,4,0)$.

(a) Find a vector equation for the plane.

The plane is parallel to both $(1,1,1)$ and $(2,6,-1) - (-1,4,0) = (3,2,-1)$.

$$\begin{aligned} \vec{x} &= \vec{x}_0 + t_1 \vec{v} + t_2(\vec{x}_1 - \vec{x}_0) \\ &= (2,6,-1) + t_1(1,1,1) + t_2(3,2,-1). \end{aligned}$$

(b) Find parametric equations for the plane.

$x = 2 + t_1 + 3t_2$, $y = 6 + t_1 + 2t_2$, $z = -1 + t_1 - t_2$.

7. (5 pts) Find the volume of the parallelepiped in $\mathbb{R}^3$ determined by the vectors $\vec{a} = (1, 0, 1)$, $\vec{b} = (1, 1, -1)$, and $\vec{c} = (0, -1, 1)$.

$$\text{volume} = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} = (0, -1, -1)$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (1, 0, 1) \cdot (0, -1, -1) = -1.$$

The volume is $\underline{1.}$

8. (5 pts) Let $\vec{u}$ and $\vec{v}$ be unit vectors in $\mathbb{R}^3$, and let $\vec{w} = \vec{u} + \vec{v}$. Show that the angle between $\vec{u}$ and $\vec{w}$ is equal to the angle between $\vec{v}$ and $\vec{w}$.

Hint: it is enough to show that the cosines of the angles are equal.

Let θ be the angle between $\vec{u}$ and $\vec{w}$.

$$\cos\theta = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \, \|\vec{w}\|} = \frac{\vec{u} \cdot (\vec{u} + \vec{v})}{\|\vec{w}\|} = \frac{\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v}}{\|\vec{w}\|} = \frac{\|\vec{u}\|^2 + \vec{u} \cdot \vec{v}}{\|\vec{w}\|} = \frac{1 + \vec{u} \cdot \vec{v}}{\|\vec{w}\|}$$

Let ϕ be the angle between $\vec{v}$ and $\vec{w}$.

$$\cos\phi = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} = \frac{\vec{v} \cdot (\vec{u} + \vec{v})}{\|\vec{w}\|} = \frac{\vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}}{\|\vec{w}\|} = \frac{\vec{u} \cdot \vec{v} + \|\vec{v}\|^2}{\|\vec{w}\|} = \frac{1 + \vec{u} \cdot \vec{v}}{\|\vec{w}\|}$$

Thus $\cos\theta = \cos\phi$, so $\theta = \phi$ (because both θ and ϕ are between 0° and 180°).