Department of Mathematics – University of Tennessee

Math 251 Matrix Algebra I

Test 1

Name: Solutions

Time allowed: 50 minutes

Instructions:

- Calculators are not allowed.
- All electronic devices must be put away.
- Answers with insufficient or incorrect working will not receive full credit.
- Simplify answers whenever possible.

Page	Points	Score
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	50	

1. (10 pts) Decide whether each statement is TRUE or FALSE. *No justification is required.*

- A linear system with more variables than equations has no solutions.

  False

- The matrix $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$ is in reduced row echelon form.

  True

- If the products AB and BA are both defined, then A and B are square matrices of the same size.

  False

- If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

  True

- The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is an elementary matrix.

  False

- If A is an $n \times n$ matrix that is not invertible, then the linear system $A\vec{x} = \vec{0}$ has infinitely many solutions.

  True

- If A is row equivalent to the identity matrix I_n, then A is invertible.

  True

- A matrix that is both upper triangular and symmetric is a diagonal matrix.

  True

- Multiplication by the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ corresponds to a counterclockwise rotation by θ.

  False

- If A is an $m \times n$ matrix, then the domain of the transformation T_A is $\mathbb{R}^m$.

  False

2. (8 pts) Use the Gauss-Jordan elimination algorithm to solve the linear system.

$$\begin{aligned} x + 2y - 2z &= 3 \\ 2x + 5y - 6z &= 8 \\ x + y + z &= 2 \end{aligned}$$

$$\left[\begin{array}{cccc} 1 & 2 & -2 & 3 \\ 2 & 5 & -6 & 8 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 2 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & -1 & 3 & -1 \end{array}\right]$$

$$\Big\downarrow R_3 \to R_3 + R_2$$

$$\left[\begin{array}{cccc} 1 & 2 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x = -3, \ y = 4, \ z = 1.$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 + 2R_3 \\ R_1 \to R_1 + 2R_3 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Big\downarrow R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

note: you were expected to follow the Gauss-Jordan algorithm for full credit.

3. (2 pts) Suppose that the augmented matrix for a linear system with variables x_1, x_2, x_3, x_4 has been reduced to the matrix below using elementary row operations. Write down the solution to the system.

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_2 = s$, $x_4 = t$. Then:

$$x_1 = 3 + 2s + t$$

$$x_3 = 5 - t .$$

4. (10 pts) Let $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$. Simplify each expression below or explain why it is not possible.

(a) $AA^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 13 \end{bmatrix}$

(b) $B^2 = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}$

(c) $BA =$ not possible.

B has 2 columns and A has 3 rows.

The dimensions do not match.

(d) $\mathrm{tr}(B + B^T) = \mathrm{tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right) = \mathrm{tr}\left(\begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix} \right) = 0 + 6 = 6.$

(e) $B^{-1} = \frac{1}{-1} \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$

5. (5 pts) How many solutions does each matrix equation have? Explain in complete sentences.

(a) $\begin{bmatrix} 4 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 4 \\ 0 \end{bmatrix}$

No solutions.

The third equation says $0=4$, so the system is inconsistent.

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -5 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -1 \\ 0 \end{bmatrix}$

One solution.

The system is consistent and there are no free variables.

(c) $\begin{bmatrix} 2 & -2 & 0 \\ 0 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \\ 0 \end{bmatrix}$

Infinite solutions.

The system is consistent and there is one free variable.

note: for full credit, your explanations should have discussed free variables and consistency.

6. (5 pts) Suppose that A is an $n \times n$ symmetric matrix.

(a) Show that A^2 is symmetric.

$$(A^2)^T = (AA)^T = A^T A^T = AA = A^2.$$

Thus A^2 is symmetric.

(b) Show that $A^2 - A + I$ is symmetric, where I is the $n \times n$ identity matrix.

$$(A^2 - A + I)^T = (A^2)^T - A^T + I^T = A^2 - A + I.$$

Thus $A^2 - A + I$ is symmetric.

7. (5 pts) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Find the inverse of A.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_2 \to R_2 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_3 \to R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) Use your answer to part (a) to solve the system $A\vec{x} = \vec{b}$.

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

8. (5 pts) The linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies

$$T_A(1,0) = (2,1,1) \qquad \text{and} \qquad T_A(2,1) = (1,3,1).$$

(a) Compute $T_A(0,1)$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ so}$$

$$T_A(0,1) = T_A(2,1) - 2T_A(1,0) = (1,3,1) - 2(2,1,1) = \underline{(-3,1,-1)}.$$

(b) What is the standard matrix A for this transformation?

$$A = \left[\, T_A(1,0) \;\middle|\; T_A(0,1) \,\right] = \underline{\begin{bmatrix} 2 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}}.$$