Department of Mathematics – University of Tennessee

Math 251 Matrix Algebra I

Test 1 Practice

Name: Solutions

Time allowed: 50 minutes

Instructions:

- Calculators are not allowed.
- All electronic devices must be put away.
- Answers with insufficient or incorrect working will not receive full credit.
- Simplify answers whenever possible.

Page	Points	Score
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	50	

1. (10 pts) Decide whether each statement is TRUE or FALSE. *No justification is required.*

- Every homogeneous linear system is consistent.

  True.

- The matrix $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is in row echelon form.

  True. (this is row echelon form, but not reduced row echelon form)

- If A and B are square matrices of the same size, then $(A+B)^2 = A^2 + 2AB + B^2$.

  False. $((A+B)^2 = A^2 + AB + BA + B^2.)$

- If AB is defined for matrices A and B, then $(AB)^T = B^T A^T$.

  True.

- If A and B are invertible matrices of the same size, then $A + B$ is invertible.

  False. (eg. $A = I$ and $B = -I$ are invertible, but $A + B = 0$ is not invertible.)

- Every elementary matrix is invertible.

  True.

- If A is invertible, and a multiple of the first row of A is added to the second row of A, then the resulting matrix is invertible.

  True. (elementary row operations do not affect invertibility)

- It is impossible for a system of linear equations to have exactly two solutions.

  True.

- The transpose of an upper triangular matrix is upper triangular.

  False. (this results in a lower triangular matrix)

- If $\vec{b}$ is a non-zero vector, then $T(\vec{x}) = \vec{x} + \vec{b}$ is a linear transformation.

  False. (T is neither homogeneous nor additive)

2. (8 pts) Use the Gauss-Jordan elimination algorithm to solve the linear system.

$$\begin{aligned} 2x + 6y - 4z &= 2 \\ y + 2z &= 1 \\ x + 5y + 2z &= 3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 6 & -4 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 5 & 2 & 3 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 5 & 2 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -8 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Let $z = t$. Then:

$$\boxed{\begin{aligned} x &= -2 + 8t, \\ y &= 1 - 2t, \\ z &= t. \end{aligned}}$$

3. (2 pts) Suppose that the augmented matrix for a linear system with variables x_1, x_2, x_3, x_4 has been reduced to the matrix below using elementary row operations. Write down the solution to the system.

$$\left[\begin{array}{ccccc} 1 & 0 & -3 & -1 & 2 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = s$, $x_4 = t$. Then:

$$\boxed{\begin{aligned} x_1 &= 2 + 3s + t \\ x_2 &= 4 - s \\ x_3 &= s \\ x_4 &= t \end{aligned}}$$

4. (10 pts) Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$. Simplify each expression below or explain why it is not possible.

(a) $A + A^T =$ impossible.

A is a 2×3 matrix and A^T is a 3×2 matrix.

Matrices can only be added if the dimensions are equal.

(b) $A^2 =$ impossible.

You cannot multiply a 2×3 matrix by a 2×3 matrix.

Matrices can only be multiplied if the number of columns in the first matrix equals the number of rows in the second matrix.

(c) $BA = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \end{bmatrix} = \underline{\begin{bmatrix} 1 & -1 & 3 \\ 2 & 2 & 2 \end{bmatrix}}.$

(d) $\operatorname{tr}(3B) = \operatorname{tr}\left(\begin{bmatrix} 3 & 0 \\ -6 & 6 \end{bmatrix}\right) = 3 + 6 = \underline{9}.$

(e) $B^{-1} = \dfrac{1}{(1)(2)-(0)(-2)}\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{bmatrix}}.$

5. (6 pts) Consider the matrix equation

$$\begin{bmatrix} 2 & -5 & 8 \\ 0 & 3 & 4 \\ 0 & 0 & k^2-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ k-1 \end{bmatrix}.$$

(a) For which value(s) of k are there an infinite number of solutions? Explain in complete sentences.

$\underline{k=1.}$ This makes the last row/equation $0=0$, which gives a consistent system with one free variable.

(b) For which value(s) of k is the system inconsistent? Explain in complete sentences.

$\underline{k=-1.}$ This makes the last row/equation $0=-2$, which gives an inconsistent system.

(c) For which value(s) of k is there a unique solution? Explain in complete sentences.

$\underline{k \neq \pm 1.}$ This gives a consistent system with no free variables.

6. (4 pts) Find the inverse of the matrix below.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_2 \to R_2 - 3R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\downarrow R_1 \to R_1 + 2R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

The inverse matrix is

$$\underline{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.}$$

7. (5 pts) Suppose that the matrix transformation T_A is defined by

$$T_A(x_1, x_2, x_3) = (2x_1 - x_3, 4x_1, x_1 + x_2 + x_3, 5x_3).$$

(a) What are the domain and codomain for the transformation?

The domain is $\underline{\mathbb{R}^3}$.

The codomain is $\underline{\mathbb{R}^4}$.

(b) What is the standard matrix A for the transformation?

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

8. (5 pts) Suppose that the matrix A satisfies the equation $A^T A = A$.

(a) Prove that A is symmetric.

$$A^T = (A^T A)^T = A^T (A^T)^T = A^T A = A$$

Because $A^T = A$, the matrix A is symmetric.

(b) Prove that $A^2 = A$.

$$A^2 = AA = A^T A = A.$$