Department of Mathematics – University of Tennessee

Math 251 Matrix Algebra I

Examination Practice

Name: ____________________

Time allowed: Two hours

Instructions:

- Calculators are not allowed.
- All electronic devices must be put away.
- Answers with insufficient or incorrect working will not receive full credit.
- Simplify answers whenever possible.

Page	Points	Score
2	10	
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10	10	
11	10	
Total:	100	

1. (10 pts) Decide whether each statement is TRUE or FALSE. *No justification is required.*

- A linear system with fewer equations than variables always has an infinite number of solutions.

- If A and B are square matrices of the same size, then $\text{tr}(AB) = \text{tr}(A)\text{tr}(B)$.

- Multiplication by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ corresponds to a rotation in two-dimensional space.

- A square matrix C is invertible if and only if $\det(C) = 0$.

- If $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$, then $\vec{v} = \vec{w}$.

- The set of all polynomials of degree 4 is a subspace of the set of all polynomials.

- The vectors $(1, -1, 1)$, $(1, 1, 1)$, and $(0, 1, 0)$ are linearly independent in $\mathbb{R}^3$.

- The column space of A is the set of all solutions to the equation $A\vec{x} = \vec{b}$.

- If B is a 3×5 matrix and $\text{rank}(B) = 2$, then $\text{nullity}(B) = 3$.

- If 0 is an eigenvalue of A, then the columns of A are linearly dependent.

2. Consider the linear system below in the variables x, y, and z.

$$\begin{aligned} x + 3y - 2z &= 1 \\ 2x + 3y + 2z &= 5 \\ x + 4y - 4z &= 0 \end{aligned}$$

(a) (7 pts) Find the row-reduced echelon form of the augmented matrix for the linear system.

(b) (3 pts) Find the general solution of the linear system.

3. The linear equations below define a transformation T_A between two vector spaces.

$$\begin{aligned} w_1 &= 3x_1 - 2x_2 + 2x_3 - x_4 \\ w_2 &= x_1 + 2x_2 + 4x_4 \\ w_3 &= 4x_1 - 2x_3 + 2x_4 \end{aligned}$$

(a) (2 pts) What are the domain and codomain for T_A?

(b) (2 pts) Compute $T_A(1, 1, 1, 1)$.

(c) (2 pts) What is the standard matrix A for this transformation?

4. (4 pts) Find the inverse of the matrix $B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 4 \\ -1 & 1 & -1 \end{bmatrix}$ or show that B is not invertible.

5. (10 pts) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & 4 \end{bmatrix}$, $\vec{u} = (1, -2, 2)$, and $\vec{v} = (2, 4, -1)$. Compute each of the following or explain why it is not possible.

(a) $\det(A) =$

(b) $AB =$

(c) $A\vec{v} =$

(d) $\vec{u} \times \vec{v} =$

(e) $\|\vec{u}\| =$

6. (5 pts) Find the determinant of the matrix $C = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \\ 4 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

7. (5 pts) Find the projection of the vector $\vec{u} = (1, 4, 2)$ onto the vector $\vec{v} = (1, 0, 1)$.

8. (4 pts) Find an equation for the plane through the point $\vec{x}_0 = (2, -3, 1)$ and orthogonal to the vector $\vec{n} = (1, 1, 1)$, and determine whether the plane contains the origin.

9. (6 pts) Let V be the set of all pairs $\vec{v} = (a, b)$ of real numbers with the following operations:

$$(a, b) + (c, d) = (a + c, 1),$$
$$k(a, b) = (k^2 a, k^2 b).$$

(a) Does V satisfy axiom 3 for vector spaces? Prove or give a counterexample.

(b) Does V satisfy axiom 9 for vector spaces? Prove or give a counterexample.

10. Consider the 2×2 matrices below.

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad M_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) (6 pts) Show that the set $\{M_1, M_2, M_3, M_4\}$ is a basis for M_{22}.

(b) (4 pts) Find the coordinates of $M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ relative to the basis $\{M_1, M_2, M_3, M_4\}$.

11. The matrices A and B below are row equivalent.

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 1 & 4 \\ 2 & 0 & 4 & 3 & 5 & 2 \\ 3 & 2 & 4 & 6 & 9 & 10 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (2 pts) Find a basis for the row space of A.

(b) (3 pts) Find a basis for the column space of A.

(c) (3 pts) Find a basis for the null space of A.

(d) (2 pts) Find the rank of A and the nullity of A.

12. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$.

(a) (3 pts) Find the characteristic polynomial of A.

(b) (2 pts) Find the eigenvalues of A.

(c) (5 pts) Find an eigenvector corresponding to each eigenvalue in part (b).

13. Suppose that the matrix B has eigenvalues $\lambda_1 = 0$ with corresponding eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\lambda_2 = 2$ with corresponding eigenvector $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(a) (4 pts) Find an invertible matrix P and a diagonal matrix D such that $B = PDP^{-1}$.

(b) (3 pts) Find the matrix B.

(c) (3 pts) Compute B^4.