Department of Mathematics – University of Tennessee

Math 251 Matrix Algebra I

Examination Practice

Name: Solutions.

Time allowed: Two hours

Instructions:

- Calculators are not allowed.
- All electronic devices must be put away.
- Answers with insufficient or incorrect working will not receive full credit.
- Simplify answers whenever possible.

Page	Points	Score
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
Total:	100	

1. (10 pts) Decide whether each statement is TRUE or FALSE. *No justification is required.*

- A linear system with fewer equations than variables always has an infinite number of solutions.

  False.

- If A and B are square matrices of the same size, then $\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$.

  False.

- Multiplication by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ corresponds to a rotation in two-dimensional space.

  False.

- A square matrix C is invertible if and only if $\det(C) = 0$.

  False.

- If $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$, then $\vec{v} = \vec{w}$.

  False.

- The set of all polynomials of degree 4 is a subspace of the set of all polynomials.

  False.

- The vectors $(1, -1, 1)$, $(1, 1, 1)$, and $(0, 1, 0)$ are linearly independent in $\mathbb{R}^3$.

  False.

- The column space of A is the set of all solutions to the equation $A\vec{x} = \vec{b}$.

  False.

- If B is a 3×5 matrix and $\mathrm{rank}(B) = 2$, then $\mathrm{nullity}(B) = 3$.

  True.

- If 0 is an eigenvalue of A, then the columns of A are linearly dependent.

  True.

2. Consider the linear system below in the variables x, y, and z.

$$\begin{aligned} x + 3y - 2z &= 1 \\ 2x + 3y + 2z &= 5 \\ x + 4y - 4z &= 0 \end{aligned}$$

(a) (7 pts) Find the row-reduced echelon form of the ~~coefficient~~ augmented matrix for the linear system.

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 2 & 3 & 2 & 5 \\ 1 & 4 & -4 & 0 \end{array}\right]$$

$$\Big\downarrow \; R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & -3 & 6 & 3 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$\Big\downarrow \; R_2 \to -\tfrac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$\Big\downarrow \; R_1 \to R_1 - R_2, \; R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow \; R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(b) (3 pts) Find the general solution of the linear system.

$x + 4z = 4$, $y - 2z = -1$.

Let $z = t$. Then $x = 4 - 4t$, $y = -1 + 2t$.

The general solution is:

$x = 4 - 4t$, $y = -1 + 2t$, $z = t$.

3. The linear equations below define a transformation T_A between two vector spaces.

$$\begin{aligned} w_1 &= 3x_1 - 2x_2 + 2x_3 - x_4 \\ w_2 &= x_1 + 2x_2 + 4x_4 \\ w_3 &= 4x_1 - 2x_3 + 2x_4 \end{aligned}$$

(a) (2 pts) What are the domain and codomain for T_A?

The domain is $\mathbb{R}^4$.

The codomain is $\mathbb{R}^3$.

(b) (2 pts) Compute $T_A(1,1,1,1)$.

$$T_A(1,1,1,1) = (3-2+2-1,\ 1+2+4,\ 4-2+2)$$
$$= (2, 7, 4).$$

(c) (2 pts) What is the standard matrix A for this transformation?

$$A = \begin{bmatrix} 3 & -2 & 2 & -1 \\ 1 & 2 & 0 & 4 \\ 4 & 0 & -2 & 2 \end{bmatrix}.$$

4. (4 pts) Find the inverse of the matrix $B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 4 \\ -1 & 1 & -1 \end{bmatrix}$ or show that B is not invertible.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & -3/2 & 1/2 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/2 & 1/2 & -3 \\ 0 & 1 & 0 & -3/2 & 1/2 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} -5/2 & 1/2 & -3 \\ -3/2 & 1/2 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

5. (10 pts) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & 4 \end{bmatrix}$, $\vec{u} = (1, -2, 2)$, and $\vec{v} = (2, 4, -1)$. Compute each of the following or explain why it is not possible.

(a) $\det(A) =$ $1 \cdot 2 \cdot 3 = 6$.

(b) $AB =$ not possible.

The number of columns of A (3) does not equal the number of rows of B (2), so AB is not defined.

(c) $A\vec{v} =$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -3 \end{bmatrix}$.

(d) $\vec{u} \times \vec{v} =$ $(1,-2,2) \times (2,4,-1) = (2-8, 4+1, 4+4)$

$= (-6, 5, 8)$.

(e) $\|\vec{u}\| =$ $\sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$.

6. (5 pts) Find the determinant of the matrix $C = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \\ 4 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

$$\det C = \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \\ 4 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{vmatrix} = 4\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 4\left((-1)\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right) - 2\left(\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}\right)$$

$$= -4(2) - 2(1+1)$$

$$= -12.$$

7. (5 pts) Find the projection of the vector $\vec{u} = (1, 4, 2)$ onto the vector $\vec{v} = (1, 0, 1)$.

$$\vec{u} \cdot \vec{v} = (1,4,2) \cdot (1,0,1) = 1 + 0 + 1 = 2$$

$$\|\vec{v}\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{2}{(\sqrt{2})^2}(1,0,1) = (1,0,1).$$

8. (4 pts) Find an equation for the plane through the point $\vec{x}_0 = (2, -3, 1)$ and orthogonal to the vector $\vec{n} = (1, 1, 1)$, and determine whether the plane contains the origin.

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \quad \Rightarrow \quad (1,1,1) \cdot (x-2, y+3, z-1) = 0$$

$$\Rightarrow \quad x - 2 + y + 3 + z - 1 = 0$$

$$\Rightarrow \quad x + y + z = 0.$$

Because $(0,0,0)$ satisfies this equation, the plane does contain the origin.

9. (6 pts) Let V be the set of all pairs $\vec{v} = (a, b)$ of real numbers with the following operations:

$$(a, b) + (c, d) = (a + c, 1),$$
$$k(a, b) = (k^2 a, k^2 b).$$

(a) Does V satisfy axiom 3 for vector spaces? Prove or give a counterexample.

<u>Yes</u>.

$$\begin{aligned} (a,b) + \big((c,d) + (e,f)\big) &= (a,b) + (c+e, 1) \\ &= (a+c+e, 1) \\ &= (a+c, 1) + (e,f) \\ &= \big((a,b) + (c,d)\big) + (e,f). \end{aligned}$$

(b) Does V satisfy axiom 9 for vector spaces? Prove or give a counterexample.

<u>Yes</u>.

$$\begin{aligned} k\big(m(a,b)\big) &= k(m^2 a, m^2 b) \\ &= (k^2 m^2 a, k^2 m^2 b) \\ &= \big((km)^2 a, (km)^2 b\big) \\ &= (km)(a,b). \end{aligned}$$

10. Consider the 2×2 matrices below.

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) (6 pts) Show that the set $\{M_1, M_2, M_3, M_4\}$ is a basis for M_{22}.

$$k_1 M_1 + k_2 M_2 + k_3 M_3 + k_4 M_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{cases} k_1 + k_2 + k_3 + k_4 = a \\ k_2 = b \\ k_3 = c \\ k_4 = d \end{cases} \Rightarrow \begin{cases} k_1 = a - b - c - d \\ k_2 = b \\ k_3 = c \\ k_4 = d \end{cases}$$

This system is consistent for any choice of a, b, c, d, so M_1, M_2, M_3, M_4 span M_{22}.

If $a = b = c = d = 0$, then $k_1 = k_2 = k_3 = k_4 = 0$, so M_1, M_2, M_3, M_4 are linearly independent.

Thus $\{M_1, M_2, M_3, M_4\}$ is a basis for M_{22}.

(b) (4 pts) Find the coordinates of $M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ relative to the basis $\{M_1, M_2, M_3, M_4\}$.

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = (2)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2M_1 - M_2 - M_3 + M_4$$

$$[M]_B = (2, -1, -1, 1).$$

(where $B = \{M_1, M_2, M_3, M_4\}$)

11. The matrices A and B below are row equivalent.

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 1 & 4 \\ 2 & 0 & 4 & 3 & 5 & 2 \\ 3 & 2 & 4 & 6 & 9 & 10 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (2 pts) Find a basis for the row space of A.

basis for row(A) = basis for row(B)

$S = \{(1,0,2,0,1,-2), (0,1,-1,0,1,2), (0,0,0,1,1,2)\}$.

(b) (3 pts) Find a basis for the column space of A.

basis for col$(B) = \{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$

basis for col$(A) = \{(1,0,2,3), (1,1,0,2), (2,1,3,6)\}$

(c) (3 pts) Find a basis for the null space of A.

$x_3 = r$, $x_5 = s$, $x_6 = t \Rightarrow x_1 = -2r - s + 2t$, $x_2 = r - s - 2t$, $x_4 = -s - 2t$

$$\begin{aligned} \vec{x} &= (-2r - s + 2t,\ r - s - 2t,\ r,\ -s - 2t,\ s,\ t) \\ &= r(-2,1,1,0,0,0) + s(-1,-1,0,-1,1,0) + t(2,-2,0,-2,0,1) \end{aligned}$$

basis for null$(A) = \{(-2,1,1,0,0,0), (-1,-1,0,-1,1,0), (2,-2,0,-2,0,1)\}$

(d) (2 pts) Find the rank of A and the nullity of A.

rank$(A) = 3$, nullity$(A) = 3$.

12. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$.

(a) (3 pts) Find the characteristic polynomial of A.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & -1 \\ -3 & \lambda \end{bmatrix}$$

$$= (\lambda - 2)\lambda - 3$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda + 1)(\lambda - 3).$$

(b) (2 pts) Find the eigenvalues of A.

$$\det(\lambda I - A) = 0 \implies \lambda = -1, \ \lambda = 3.$$

(c) (5 pts) Find an eigenvector corresponding to each eigenvalue in part (b).

$\underline{\lambda = -1:}$ $\begin{bmatrix} -3 & -1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = t, \ y = -3t$

eigenvector: $\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

$\underline{\lambda = 3:}$ $\begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = t, \ y = t$

eigenvector: $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

13. Suppose that the matrix B has eigenvalues $\lambda_1 = 0$ with corresponding eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\lambda_2 = 2$ with corresponding eigenvector $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(a) (4 pts) Find an invertible matrix P and a diagonal matrix D such that $B = PDP^{-1}$.

$$P = \left[\vec{v}_1 \,\middle|\, \vec{v}_2\right] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

~~check:~~ $\cancel{PDP^{-1}}$

(b) (3 pts) Find the matrix B.

$$B = PDP^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix}.$$

(c) (3 pts) Compute B^4.

$$B^4 = P(D^4)P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 16 & 16 \end{bmatrix}$$
$$= \begin{bmatrix} -16 & -16 \\ 32 & 32 \end{bmatrix}.$$