Name: Solutions.

1. Solve the system by inverting the coefficient matrix and using the following theorem:

Theorem. If A is an invertible matrix, then the system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

$$\begin{aligned} x + y + z &= 5 \\ x + y - 4z &= 10 \\ -4x + y + z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 4R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \end{array}\right]$$

$\downarrow$ $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right]$$

$\downarrow$ $R_2 \to \frac{1}{5}R_2$, $R_3 \to -\frac{1}{5}R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4/5 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$\downarrow$ $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 4/5 & 1/5 & 0 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$\downarrow$ $R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -1/5 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & -\frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & 0 \end{bmatrix},$$ so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & -\frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}.$$