Section 5.1 Eigenvalues and Eigenvectors

"eigen" = "own"

Objectives.

- Introduce eigenvalues and eigenvectors for a matrix or matrix transformation.
- Find eigenvalues, eigenvectors, and eigenspaces.

Suppose that $\vec{x}$ is a non-zero vector and A is a square matrix. If $A\vec{x} = \lambda\vec{x}$ for some scalar λ, then λ is an eigenvalue of A and $\vec{x}$ is an eigenvector of A corresponding to λ.

Example 1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Compute $A\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Which of these vectors is an eigenvector of A? What are the eigenvalues?

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ i.e. $\lambda = 1$.

Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 1$.

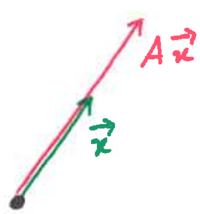
$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ i.e. $\lambda = 2$.

Thus $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 2$.

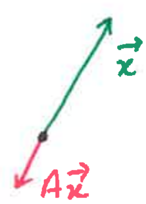
$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \lambda\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not an eigenvector of A.

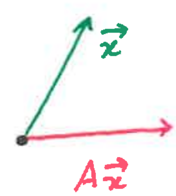
Loosely speaking, an eigenvector of an $n \times n$ matrix A (or of the matrix operator T_A) is a direction in $\mathbb{R}^n$ that is unchanged when multiplying by A. That is, $\vec{x} \neq \vec{0}$ is an eigenvector of A if $\vec{x}$ and $A\vec{x}$ are parallel.



$\vec{x}$ is an eigenvector of A with $\lambda > 1$.



$\vec{x}$ is an eigenvector of A with $-1 < \lambda < 0$.



$\vec{x}$ is not an eigenvector of A.
(b/c $A\vec{x} \neq \lambda\vec{x}$)

characteristic equation of A.

Theorem. If A is a square matrix, then λ is an eigenvalue of A if and only if $\det(\lambda I - A) = 0$.

Proof. Suppose λ is an eigenvalue of A. Then there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. That is, $A\vec{x} = \lambda I\vec{x}$, so $\vec{0} = \lambda I\vec{x} - A\vec{x} = (\lambda I - A)\vec{x}$. Thus $\det(\lambda I - A) = 0$.

Suppose $\det(\lambda I - A) = 0$. Then there is a nonzero vector $\vec{x}$ such that $(\lambda I - A)\vec{x} = \vec{0}$. Thus $\lambda I\vec{x} - A\vec{x} = \vec{0}$, so $A\vec{x} = \lambda I\vec{x} = \lambda\vec{x}$. Therefore, λ is an eigenvalue of A.

Example 2. Use the theorem above to find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda-2 \end{bmatrix}.$$

characteristic polynomial of A.

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda-2 \end{bmatrix} = (\lambda-1)(\lambda-2) - (-1)(0) = (\lambda-1)(\lambda-2).$$

- solve $\det(\lambda I - A) = 0$:

$$(\lambda-1)(\lambda-2) = 0 \implies \lambda = 1, 2.$$

The eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

Strategy. To find the eigenvalues of A:

- set up the characteristic equation/polynomial of A
- find all the solutions of the characteristic equations.

Example 3. Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$.

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & -2 & 0 \\ -3 & \lambda-1 & -2 \\ 0 & -3 & \lambda-1 \end{bmatrix} = (\lambda-1)\left((\lambda-1)^2-6\right) - (-2)\left((-3)(\lambda-1)\right)$$

$$= (\lambda-1)\left((\lambda-1)^2-6-6\right) = \underbrace{(\lambda-1)(\lambda^2-2\lambda-11)}_{\text{characteristic polynomial}}.$$

$\det(\lambda I - A) = 0 \implies \lambda = 1$ or $\lambda^2 - 2\lambda - 11 = 0$ ← use quadratic formula!!!

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4+44}}{2} = 1 \pm \frac{\sqrt{48}}{2} = 1 \pm 2\sqrt{3}.$$

The eigenvalues are $\lambda = 1$, $\lambda = 1 + 2\sqrt{3}$, $\lambda = 1 - 2\sqrt{3}$.

The eigenvalues of a triangular matrix can be found 'by inspection' (that is, without solving the characteristic polynomial).

Theorem. If A is triangular, then the eigenvalues of A are the entries on the main diagonal.

Example 4. Find the eigenvalues of each matrix.

$$\begin{bmatrix} 3 & 9 & -4 \\ 0 & -7 & 5 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 1 & -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

$\lambda = 3, -7, 4$. $\qquad \lambda = \frac{1}{2}, \frac{3}{2}, 2$. $\qquad \lambda = a, b, c, d$.

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-3 & -9 & 4 \\ 0 & \lambda+7 & -5 \\ 0 & 0 & \lambda-4 \end{bmatrix} = (\lambda-3)(\lambda+7)(\lambda-4).$$

> **Theorem.** If A is a square matrix, then the following statements are equivalent.
>
> 1. λ is an eigenvalue of A.
> 2. λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.
> 3. The system $(\lambda I - A)\vec{x} = \vec{0}$ has nontrivial solutions.
> 4. There is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$.

Now that we know how to find eigenvalues for a matrix, we turn our attention to finding the eigenvectors corresponding to each eigenvalue. If λ is an eigenvalue of A, then the eigenvectors corresponding to λ are the nonzero vectors $\vec{x}$ such that $(\lambda I - A)\vec{x} = \vec{0}$. This solution space is the eigenspace corresponding to λ.

- find all eigenvalues of A
- solve $(\lambda I - A)\vec{x} = \vec{0}$ for each eigenvalue λ.

Example 5. Find the eigenspaces of the matrix $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$.

$\det(\lambda I - A) = \det\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix} = (\lambda+1)\lambda - 6 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$.

The eigenvalues of A are $\lambda = -3$, $\lambda = 2$.

$\lambda = 2$: $\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\Longrightarrow$ $x_1 = t$, $x_2 = t$. use "elimination" to solve.

Thus $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to $\lambda = 2$.

$\lambda = -3$: $\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Longrightarrow$ $x_1 = -\frac{3}{2}t$, $x_2 = t$.

Thus $\left\{\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to $\lambda = -3$.

cofactor expansion!!!

Example 6. Find the eigenspaces of the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} = \cdots = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda-1)(\lambda-2)^2.$$

The eigenvalues of A are $\lambda=1$ and $\lambda=2$. ← repeated eigenvalue.

$\underline{\lambda=1:}$ $\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -2s,\ x_2 = s,\ x_3 = s.$

The ~~eigenvalue~~ eigenvectors for $\lambda=1$ are $s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, so $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for this eigenspace.

$\underline{\lambda=2:}$ $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -t,\ x_2 = s,\ x_3 = t$

The eigenvectors for $\lambda=2$ are $\begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, so $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for this eigenspace.

Theorem. The square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Example 7. Find the eigenvalues of $A = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$.

$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & 3 \\ 0 & \lambda \end{bmatrix} = (\lambda-1)\lambda$, so $\lambda = 0, 1$ are the eigenvalues of A. Thus A is <u>not</u> invertible.