Section 4.9 Rank, Nullity, and the Fundamental Matrix Spaces

Objectives.

- Define the rank and nullity of a matrix, and see how these are related.
- Introduce the orthogonal complement of a subspace.
- Extend the Equivalence Theorem.

Recall the following definitions from Section 4.8.

- the row space of A is the set of all linear combinations of the row vectors of A
- the column space of A is the set of all linear combinations of the column vectors of A
- the null space of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$

The dimensions of these three spaces are related, and depend on the number of "leading variables" and "free variables" in a linear system.

Theorem. The row space and column space of a matrix A have the same dimension.

The common dimension of the row space and the column space of A is called the rank of A. The dimension of the null space of a matrix A is called the nullity of A.

Example 1. What is the rank of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$? What is the nullity of A?

$\{(1,0),(0,1)\}$ is a basis for row(A), so rank$(A) = 2$.

(also, $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A).)

The only vector in null(A) is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so nullity$(A) = 0$.

i.e. null(A) is the zero vector space.

Theorem. If A is an $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = n$.

number of columns.

We can also relate the rank and nullity of a matrix with the number of leading variables and the number of free variables in a homogeneous linear system.

Theorem. Let A be an $m \times n$ matrix. Then $\text{rank}(A)$ is the number of leading variables in the general solution to $A\vec{x} = \vec{0}$, and $\text{nullity}(A)$ is the number of free variables in the general solution to $A\vec{x} = \vec{0}$.

Example 2. The matrices A, B, and C below are row equivalent.

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 1 & 2 & 1 & 0 & 2 \\ 2 & 4 & 2 & 1 & 5 \\ 1 & 0 & 3 & -2 & -2 \end{bmatrix} \xrightarrow{\text{ref}} B = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} C = \begin{bmatrix} 1 & 0 & \boxed{3} & 0 & \boxed{0} \\ 0 & 1 & \boxed{-1} & 0 & \boxed{1} \\ 0 & 0 & \boxed{0} & 1 & \boxed{1} \\ 0 & 0 & \boxed{0} & 0 & \boxed{0} \end{bmatrix}$$

$x_3 = s \quad x_5 = t$

(a) Find a basis for row(A).

- row(A) = row(B) = row(C) because A, B, C are row equivalent.

basis for row$(A) = \{(1,1,2,-1,0), (0,1,-1,1,2), (0,0,0,1,1)\}$.

(b) Find a basis for col(A).

$$\text{basis for col}(B) = \left\{ \begin{array}{c} \text{col } 1 \\ \downarrow \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} \text{col } 2 \\ \downarrow \\ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} \text{col } 4 \\ \downarrow \\ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{array} \right\}, \text{ so}$$

use the corresponding columns of A.

$$\text{basis for col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

(c) What is the rank of A?

rank$(A) = 3$

because ... dim(row(A)) = 3 or dim(col(A)) = 3 or A has 3 leading variables.

(d) Find a basis for null(A).

- sol$^{\underline{n}}$ to $A\vec{x} = \vec{0}$ is $x_3 = s$, $x_5 = t$, $x_1 = -3s$, $x_2 = s - t$, $x_4 = -t$,

or $\vec{x} = (-3s, s-t, s, -t, t) = s(-3,1,1,0,0) + t(0,-1,0,-1,1)$.

basis for null$(A) = \{(-3,1,1,0,0), (0,-1,0,-1,1)\}$.

(e) What is the nullity of A?

nullity$(A) = 2$

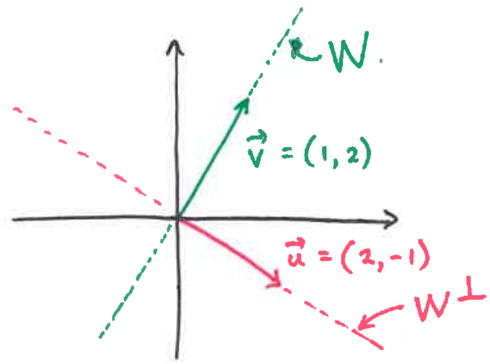
because ... dim(null(A)) = 2 or A has 2 free variables or $n - \text{rank}(A) = 5 - 3 = \underline{2}$.

If W is a subspace of $\mathbb{R}^n$, then the set of all vectors in $\mathbb{R}^n$ that are orthogonal to *every* vector in W is called the orthogonal complement of W, and is denoted by $W^\perp$. ← "W perp"

Example 3. Let $W = \text{span}\{(1,2)\}$, which is a subspace of $\mathbb{R}^2$.

(a) Find a vector in $W^\perp$. $\vec{u} = (2,-1)$.

(why? if $\vec{v}$ is in W, then $\vec{v} = k(1,2) = (k,2k)$.
Thus $(2,-1)\cdot(k,2k) = 2k-2k = 0$.)



(b) Describe the set of all vectors in $W^\perp$.

$W^\perp$ contains all vector parallel to $(2,-1)$.
(why? $(2\ell,-\ell)\cdot(k,2k) = 2k\ell - 2k\ell = 0$.)

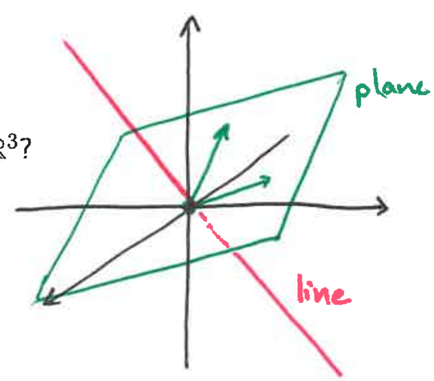
note: $\{\vec{0}\}$ is the orthogonal complement of $\mathbb{R}^2$ in $\mathbb{R}^2$.

Theorem. If W is a subspace of $\mathbb{R}^n$, then:

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.
2. The only vector in both W and $W^\perp$ is $\vec{0}$.
3. The orthogonal complement of $W^\perp$ is W.

Example 4. (a) What is the orthogonal complement of a line through the origin in $\mathbb{R}^3$?

a plane through the origin.
(i.e. the plane that is orthogonal to any vector on the line)



(b) What is the orthogonal complement of a plane through the origin in $\mathbb{R}^3$?

a line through the origin.
(i.e. the line that is orthogonal to any vector on the plane)

Recall that if $\vec{x}_h$ is a solution to the homogeneous linear system $A\vec{x} = \vec{0}$, then $\vec{x}_h$ is orthogonal to every row of A. That is, $\vec{x}_h \cdot \vec{r}_i = 0$ where $\vec{r}_i$ is the ith row of A.

Theorem. If A is an $m \times n$ matrix, then:

1. The null space of A and the row space of A are orthogonal complements in $\mathbb{R}^n$.
2. The null space of A^T and the column space of A are orthogonal complements in $\mathbb{R}^m$.

Example 5. Let $\vec{x}_h$ be a solution to the homogeneous linear system $A\vec{x} = \vec{0}$, and let $\vec{r}$ be a vector in the row space of A. Show that $\vec{x}_h$ is orthogonal to $\vec{r}$.

Because $\vec{r}$ is in the row space of A, we can write

$$\vec{r} = c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_m\vec{r}_m$$

where $\vec{r}_i$ is the i^{th} row of A.

Then:

$$\begin{aligned}
\vec{x}_h \cdot \vec{r} &= \vec{x}_h \cdot (c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_m\vec{r}_m) \\
&= \vec{x}_h \cdot (c_1\vec{r}_1) + \vec{x}_h \cdot (c_2\vec{r}_2) + \cdots + \vec{x}_h \cdot (c_m\vec{r}_m) \\
&= c_1\, \vec{x}_h \cdot \vec{r}_1 + c_2\, \vec{x}_h \cdot \vec{r}_2 + \cdots + c_m\, \vec{x}_h \cdot \vec{r}_m \\
&= c_1(0) + c_2(0) + \cdots + c_m(0) \\
&= 0.
\end{aligned}$$

That is, $\vec{x}_h$ is orthogonal to $\vec{r}$.

<u>note</u>: This proves part (1) of the theorem above, because we have shown that any vector in null(A) is orthogonal to any vector in row(A).

We finally have all the ingredients to state the "Equivalence Theorem" in full.

Equivalence Theorem. If A is an $n \times n$ matrix with no repeated rows or repeated columns, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.
5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.
7. $\det A \neq 0$.
8. The column vectors of A are linearly independent.
9. The row vectors of A are linearly independent.
10. The column vectors of A span $\mathbb{R}^n$.
11. The row vectors of A span $\mathbb{R}^n$.
12. The column vectors of A are a basis for $\mathbb{R}^n$.
13. The row vectors of A are a basis for $\mathbb{R}^n$.
14. $\text{rank}(A) = n$.
15. $\text{nullity}(A) = 0$.
16. The orthogonal complement of $\text{null}(A)$ is $\mathbb{R}^n$.
17. The orthogonal complement of $\text{row}(A)$ is $\{\vec{0}\}$.