Section 4.8 Row Space, Column Space, Null Space

Objectives.

- Introduce the row space, column space, and null space for a matrix.
- Study how solutions to homogeneous and nonhomogeneous systems are related.
- Find a basis and the dimension of the row space, column space, and null space.

Given an $m \times n$ matrix A, we an define three natural subspaces of Euclidean space.

- the row space of A is the set of all linear combinations of the row vectors of A

  Question: *Is the row space of A a subspace of $\mathbb{R}^m$ or of $\mathbb{R}^n$?*

  - row vectors in A have length n.

- the column space of A is the set of all linear combinations of the column vectors of A

  Question: *Is the column space of A a subspace of $\mathbb{R}^m$ or of $\mathbb{R}^n$?*

  - column vectors in A have length m.

- the null space of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$

  Question: *Is the null space of A a subspace of $\mathbb{R}^m$ or of $\mathbb{R}^n$?*

  - if $A\vec{x}$ is defined, then $\vec{x}$ has length n.

Example 1. Let $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \\ 1 & 3 \end{bmatrix}$.

(a) The set row(A) (the row space of A) is a subspace of $\underline{\mathbb{R}^2}$.

(b) Name one vector in row(A). $[2 \;\; 1]$ (or $[4 \;\; -1]$, $[1 \;\; 3]$, $[6 \;\; 0]$, ...)

(c) The set col(A) (the column space of A) is a subspace of $\underline{\mathbb{R}^3}$.

(d) Name one vector in col(A). $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$, ...)

(e) The set null(A) (the null space of A) is a subspace of $\underline{\mathbb{R}^2}$.

(f) Name one vector in null(A). $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (this is the only vector in null(A)).

Example 2. Consider the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Is $(2, -2, 2)$ in $\mathrm{row}(A)$? No.

if $k_1(1,0,-1) + k_2(0,1,1) = (2,-2,2)$, then $k_1 = 2$ and $k_2 = -2$, but
$2(1,0,-1) + (-2)(0,1,1) = (2,-2,0) \neq (2,-2,2)$.

(b) What is a basis for $\mathrm{row}(A)$?

$$S = \{(1,0,-1), (0,1,1)\}.$$

(c) What is the dimension of $\mathrm{row}(A)$?

the dimension is the number of vectors in a basis.

$$\dim(\mathrm{row}(A)) = 2.$$

(d) Is $(4, 2)$ in $\mathrm{col}(A)$? Yes.

$$4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

(e) What is a basis for $\mathrm{col}(A)$?

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

note: $\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the columns of A are not linearly independent.

(f) What is the dimension of $\mathrm{col}(A)$?

$$\dim(\mathrm{col}(A)) = 2.$$

(g) Is $(2, -2, 2)$ in $\mathrm{null}(A)$? Yes.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(h) What is a basis for $\mathrm{null}(A)$?

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

note: every vector in $\mathrm{null}(A)$ is a multiple of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(i) What is the dimension of $\mathrm{null}(A)$?

$$\dim(\mathrm{null}(A)) = 1.$$

The column space of a matrix can also be described as the set of all vectors $\vec{b}$ in $\mathbb{R}^n$ for which the equation $A\vec{x} = \vec{b}$ has a solution.

Theorem. The equation $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in the column space of A.

Example 3. Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -3 \\ -2 \\ 6 \end{bmatrix}.$$

Show that $\vec{b}$ is in the column space of A.

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ -1 & 3 & 1 & -2 \\ 2 & 2 & 1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 3 & -5 \\ 0 & 6 & 3 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & -21 & 42 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Thus $3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 6 \end{bmatrix}.$

Example 4. Suppose that $\vec{x}_h$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$, and $\vec{x}_0$ is a solution of the nonhomogeneous system $A\vec{x} = \vec{b}$. Show that $\vec{x}_0 + k\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$ for all scalars k.

$A\vec{x}_0 = \vec{b}$ and $A\vec{x}_h = \vec{0}$, so

$$A(\vec{x}_0 + k\vec{x}_h) = A\vec{x}_0 + A(k\vec{x}_h) = \vec{b} + k(A\vec{x}_h) = \vec{b} + k(\vec{0}) = \vec{b}.$$

The importance of the last example is the following principle:

The general solution for a consistent linear system is the sum of a particular solution for the linear system and the general solution for the corresponding homogeneous linear system.

Theorem. Every solution $\vec{x}$ for a consistent linear system $A\vec{x} = \vec{b}$ can be written in the form

$$\vec{x} = \vec{x}_0 + c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r,$$

where $\vec{x}_0$ is any solution for $A\vec{x} = \vec{b}$ and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is a basis for the null space of A.

Finding a basis for the row space or column space of a matrix.

Recall that two matrices are row equivalent if each can be obtained from the other through elementary row operations.

Theorem.

1. If A and B are row equivalent, then $\text{row}(A) = \text{row}(B)$.

2. If A and B are row equivalent, then $\text{null}(A) = \text{null}(B)$.

For a matrix A in row-echelon form (such as in Example 2), identifying a basis for $\text{row}(A)$ or $\text{col}(A)$ is particularly easy – the row vectors containing a leading 1 form a basis for $\text{row}(A)$, and the column vectors containing a leading 1 form a basis for $\text{col}(A)$.

Example 5. Find a basis for $\text{row}(B)$ and a basis for $\text{col}(B)$ given that $B = \begin{bmatrix} 1 & -3 & 0 & 4 & -1 \\ 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

basis for row(B): $S = \{(1,-3,0,4,-1), (0,1,2,-2,0), (0,0,0,1,1)\}$.

basis for col(B): $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

More generally, a basis for $\text{row}(A)$ can be found by reducing A to ref and applying the theorem above.

Example 6. Find a basis for $\text{row}(A)$ given that $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -4/3 \end{bmatrix}.$$

$S = \{(1,2,-1,3), (0,1,2,-3), (0,0,1,-4/3)\}$ is a basis for row(A).

The next theorem allows us to find a basis for $\text{col}(A)$ – more specifically, a basis for $\text{col}(A)$ that consists entirely of columns of A.

Theorem. Suppose that A and B are row equivalent.

1. If a set of columns of A are linearly independent, then the corresponding columns of B are also linearly independent.
2. If a set of columns of A are a basis for $\text{col}(A)$, then the corresponding columns of B are a basis for $\text{col}(B)$.

Example 7. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 3 & 2 & -1 & 0 & 2 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix}$.

(a) Find a matrix B in row-echelon form that is row equivalent to A.

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 3 & 2 & -1 & 0 & 2 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & -1 & 5 & -3 & -10 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & -1 & 5 & -3 & -10 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & 1 & -5 & 3 & 10 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Identify a basis for $\text{col}(B)$ in part (a).

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix} \right\}.$$

the columns of B that contain a "leading 1" form a basis for $\text{col}(B)$.

(c) Use the theorem above to identify a basis for $\text{col}(A)$ that consists entirely of columns of A.

- because columns 1,2,5 are a basis for $\text{col}(B)$, the corresponding columns of A are a basis for $\text{col}(A)$.

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

(d) What is the dimension of $\text{col}(A)$?

$$\dim(\text{col}(A)) = 3.$$

Suppose that we want to find a basis for $\mathrm{row}(A)$ that consists entirely of rows of A. One way to do this is to apply the method from the previous page to the matrix A^T. This gives a basis for $\mathrm{col}(A^T)$ that consists of columns of A^T – transposing this basis gives a basis for $\mathrm{row}(A)$ that consists of rows of A.

Example 8. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 3 & 2 & -1 & 0 & 2 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix}$ from Example 7.

(a) Find a basis for $\mathrm{col}(A^T)$ that consists entirely of columns of A^T.

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & -1 & 5 \\ 1 & 0 & -3 \\ 4 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & -1 \\ 0 & 5 & 5 \\ 0 & -3 & -3 \\ 0 & -10 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \\ 0 & -3 & -3 \\ 0 & -10 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because all three columns of the reduced matrix contain a leading 1, we need all three columns of A^T in a basis for $\mathrm{col}(A^T)$.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 5 \\ -3 \\ -2 \end{bmatrix} \right\}.$$

(b) Find a basis for $\mathrm{row}(A)$ that consists entirely of rows of A.

$$S = \{ (1,1,-2,1,4), (3,2,-1,0,2), (0,-1,5,-3,-2) \}.$$

(c) What is the dimension of $\mathrm{row}(A)$? $\dim(\mathrm{row}(A)) = 3$.