## Section 4.8 Row Space, Column Space, Null Space

**Objectives.**

- Introduce the row space, column space, and null space for a matrix.
- Study how solutions to homogeneous and nonhomogeneous systems are related.
- Find a basis and the dimension of the row space, column space, and null space.

---

Given an $m \times n$ matrix $A$, we an define three natural subspaces of Euclidean space.

- the row space of $A$ is the set of all linear combinations of the row vectors of $A$

  **Question:** *Is the row space of $A$ a subspace of $\mathbb{R}^m$ or of $\mathbb{R}^n$?*

  - row vectors in $A$ have length $n$.

- the column space of $A$ is the set of all linear combinations of the column vectors of $A$

  **Question:** *Is the column space of $A$ a subspace of $\mathbb{R}^m$ or of $\mathbb{R}^n$?*

  - column vectors in $A$ have length $m$.

- the null space of $A$ is the set of all solutions to the equation $A\vec{x} = \vec{0}$

  **Question:** *Is the null space of $A$ a subspace of $\mathbb{R}^m$ or of $\mathbb{R}^n$?*

  - if $A\vec{x}$ is defined, then $\vec{x}$ has length $n$.

**Example 1.** Let $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \\ 1 & 3 \end{bmatrix}$.

(a) The set $\text{row}(A)$ (the row space of $A$) is a subspace of $\mathbb{R}^2$.

(b) Name one vector in $\text{row}(A)$. $[2 \quad 1]$ (or $[4 \ -1]$, $[1 \quad 3]$, $[6 \quad 0]$, ...)

(c) The set $\text{col}(A)$ (the column space of $A$) is a subspace of $\mathbb{R}^3$.

(d) Name one vector in $\text{col}(A)$. $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$, ...)

(e) The set $\text{null}(A)$ (the null space of $A$) is a subspace of $\mathbb{R}^2$.

(f) Name one vector in $\text{null}(A)$. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (this is the only vector in $\text{null}(A)$).

**Example 2.** Consider the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Is $(2, -2, 2)$ in $\operatorname{row}(A)$? No.

if $k_1(1,0,-1) + k_2(0,1,1) = (2,-2,2)$, then $k_1 = 2$ and $k_2 = -2$, but
$2(1,0,-1) + (-2)(0,1,1) = (2,-2,0) \neq (2,-2,2)$.

(b) What is a basis for $\operatorname{row}(A)$?

$$S = \{(1,0,-1), (0,1,1)\}.$$

(c) What is the dimension of $\operatorname{row}(A)$?

the dimension is the number of vectors in a basis.

$$\dim(\operatorname{row}(A)) = 2.$$

(d) Is $(4, 2)$ in $\operatorname{col}(A)$? Yes.

$$4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

(e) What is a basis for $\operatorname{col}(A)$?

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

note: $\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the columns of $A$ are not linearly independent.

(f) What is the dimension of $\operatorname{col}(A)$?

$$\dim(\operatorname{col}(A)) = 2.$$

(g) Is $(2, -2, 2)$ in $\operatorname{null}(A)$? Yes.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(h) What is a basis for $\operatorname{null}(A)$?

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

note: every vector in $\operatorname{null}(A)$ is a multiple of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(i) What is the dimension of $\operatorname{null}(A)$?

$$\dim(\operatorname{null}(A)) = 1.$$

The column space of a matrix can also be described as the set of all vectors $\vec{b}$ in $\mathbb{R}^n$ for which the equation $A\vec{x} = \vec{b}$ has a solution.

**Theorem.** The equation $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in the column space of $A$.

**Example 3.** Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -3 \\ -2 \\ 6 \end{bmatrix}.$$

Show that $\vec{b}$ is in the column space of $A$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ -1 & 3 & 1 & -2 \\ 2 & 2 & 1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 3 & -5 \\ 0 & 6 & 3 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & -21 & 42 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Thus $3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 6 \end{bmatrix}$.

**Example 4.** Suppose that $\vec{x}_h$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$, and $\vec{x}_0$ is a solution of the nonhomogeneous system $A\vec{x} = \vec{b}$. Show that $\vec{x}_0 + k\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$ for all scalars $k$.

$A\vec{x}_0 = \vec{b}$ and $A\vec{x}_h = \vec{0}$, so

$$A(\vec{x}_0 + k\vec{x}_h) = A\vec{x}_0 + A(k\vec{x}_h) = \vec{b} + k(A\vec{x}_h) = \vec{b} + k(\vec{0}) = \vec{b}.$$

The importance of the last example is the following principle:

*The general solution for a consistent linear system is the sum of a particular solution for the linear system and the general solution for the corresponding homogeneous linear system.*

**Theorem.** Every solution $\vec{x}$ for a consistent linear system $A\vec{x} = \vec{b}$ can be written in the form

$$\vec{x} = \vec{x}_0 + c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r,$$

where $\vec{x}_0$ is any solution for $A\vec{x} = \vec{b}$ and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is a basis for the null space of $A$.

**Finding a basis for the row space or column space of a matrix.**

Recall that two matrices are row equivalent if each can be obtained from the other through elementary row operations.

**Theorem.**

1. If $A$ and $B$ are row equivalent, then $\text{row}(A) = \text{row}(B)$.
2. If $A$ and $B$ are row equivalent, then $\text{null}(A) = \text{null}(B)$.

For a matrix $A$ in row-echelon form (such as in Example 2), identifying a basis for $\text{row}(A)$ or $\text{col}(A)$ is particularly easy – the row vectors containing a leading 1 form a basis for $\text{row}(A)$, and the column vectors containing a leading 1 form a basis for $\text{col}(A)$.

**Example 5.** Find a basis for $\text{row}(B)$ and a basis for $\text{col}(B)$ given that $B = \begin{bmatrix} 1 & -3 & 0 & 4 & -1 \\ 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

basis for $\text{row}(B)$: $S = \{(1,-3,0,4,-1), (0,1,2,-2,0), (0,0,0,1,1)\}$.

basis for $\text{col}(B)$: $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

More generally, a basis for $\text{row}(A)$ can be found by reducing $A$ to ref and applying the theorem above.

**Example 6.** Find a basis for $\text{row}(A)$ given that $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -4/3 \end{bmatrix}.$$

$S = \{(1,2,-1,3), (0,1,2,-3), (0,0,1,-4/3)\}$ is a basis for $\text{row}(A)$.

The next theorem allows us to find a basis for $\mathrm{col}(A)$ – more specifically, a basis for $\mathrm{col}(A)$ that consists entirely of columns of $A$.

> **Theorem.** Suppose that $A$ and $B$ are row equivalent.
>
> 1. If a set of columns of $A$ are linearly independent, then the corresponding columns of $B$ are also linearly independent.
>
> 2. If a set of columns of $A$ are a basis for $\mathrm{col}(A)$, then the corresponding columns of $B$ are a basis for $\mathrm{col}(B)$.

**Example 7.** Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 3 & 2 & -1 & 0 & 2 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix}$.

(a) Find a matrix $B$ in row-echelon form that is row equivalent to $A$.

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 3 & 2 & -1 & 0 & 2 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & -1 & 5 & -3 & -10 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & -1 & 5 & -3 & -10 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & 1 & -5 & 3 & 10 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Identify a basis for $\mathrm{col}(B)$ in part (a).

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix} \right\}.$$

the columns of $B$ that contain a "leading 1" form a basis for $\mathrm{col}(B)$.

(c) Use the theorem above to identify a basis for $\mathrm{col}(A)$ that consists entirely of columns of $A$.

- because columns 1,2,5 are a basis for $\mathrm{col}(B)$, the corresponding columns of $A$ are a basis for $\mathrm{col}(A)$.

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

(d) What is the dimension of $\mathrm{col}(A)$?

$$\dim(\mathrm{col}(A)) = 3.$$

Suppose that we want to find a basis for $\text{row}(A)$ that consists entirely of rows of $A$. One way to do this is to apply the method from the previous page to the matrix $A^T$. This gives a basis for $\text{col}(A^T)$ that consists of columns of $A^T$ – transposing this basis gives a basis for $\text{row}(A)$ that consists of rows of $A$.

**Example 8.** Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 3 & 2 & -1 & 0 & 2 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix}$ from Example 7.

(a) Find a basis for $\text{col}(A^T)$ that consists entirely of columns of $A^T$.

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & -1 & 5 \\ 1 & 0 & -3 \\ 4 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & -1 \\ 0 & 5 & 5 \\ 0 & -3 & -3 \\ 0 & -10 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \\ 0 & -3 & -3 \\ 0 & -10 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because all three columns of the reduced matrix contain a leading 1, we need all three columns of $A^T$ in a basis for $\text{col}(A^T)$.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 5 \\ -3 \\ -2 \end{bmatrix} \right\}.$$

(b) Find a basis for $\text{row}(A)$ that consists entirely of rows of $A$.

$$S = \{ (1,1,-2,1,4), (3,2,-1,0,2), (0,-1,5,-3,-2) \}.$$

(c) What is the dimension of $\text{row}(A)$? $\dim(\text{row}(A)) = 3.$

## Section 4.9 Rank, Nullity, and the Fundamental Matrix Spaces

**Objectives.**

- Define the rank and nullity of a matrix, and see how these are related.
- Introduce the orthogonal complement of a subspace.
- Extend the Equivalence Theorem.

---

Recall the following definitions from Section 4.8.

- the row space of $A$ is the set of all linear combinations of the row vectors of $A$
- the column space of $A$ is the set of all linear combinations of the column vectors of $A$
- the null space of $A$ is the set of all solutions to the equation $A\vec{x} = \vec{0}$

The dimensions of these three spaces are related, and depend on the number of "leading variables" and "free variables" in a linear system.

**Theorem.** The row space and column space of a matrix $A$ have the same dimension.

The common dimension of the row space and the column space of $A$ is called the rank of $A$. The dimension of the null space of a matrix $A$ is called the nullity of $A$.

**Example 1.** What is the rank of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$? What is the nullity of $A$?

$\{(1,0),(0,1)\}$ is a basis for row$(A)$, so rank$(A) = 2$.

(also, $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ is a basis for col$(A)$.)

The only vector in null$(A)$ is $\begin{bmatrix}0\\0\end{bmatrix}$, so nullity$(A) = 0$.

i.e. null$(A)$ is the zero vector space.

**Theorem.** If $A$ is an $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = n$.

number of columns.

We can also relate the rank and nullity of a matrix with the number of leading variables and the number of free variables in a homogeneous linear system.

**Theorem.** Let $A$ be an $m \times n$ matrix. Then $\text{rank}(A)$ is the number of leading variables in the general solution to $A\vec{x} = \vec{0}$, and $\text{nullity}(A)$ is the number of free variables in the general solution to $A\vec{x} = \vec{0}$.

**Example 2.** The matrices $A$, $B$, and $C$ below are row equivalent.

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 1 & 2 & 1 & 0 & 2 \\ 2 & 4 & 2 & 1 & 5 \\ 1 & 0 & 3 & -2 & -2 \end{bmatrix} \xrightarrow[\text{ref}]{} B = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{rref}]{} C = \begin{bmatrix} 1 & 0 & \boxed{3} & 0 & \boxed{0} \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3 = s \quad x_5 = t$

(a) Find a basis for row$(A)$.

- row$(A)$ = row$(B)$ = row$(C)$ because $A, B, C$ are row equivalent.

basis for row$(A) = \{(1,1,2,-1,0), (0,1,-1,1,2), (0,0,0,1,1)\}$.

(b) Find a basis for col$(A)$.

$$\begin{array}{ccc} \text{col } 1 & \text{col } 2 & \text{col } 4 \end{array}$$

basis for col$(B) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, so use the corresponding columns of $A$.

basis for col$(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$.

(c) What is the rank of $A$?

rank$(A) = 3$ because ... dim(row$(A)$) = 3 or dim(col$(A)$) = 3 or $A$ has 3 leading variables.

(d) Find a basis for null$(A)$.

- sol$^{\text{n}}$ to $A\vec{x} = \vec{0}$ is $x_3 = s$, $x_5 = t$, $x_1 = -3s$, $x_2 = s - t$, $x_4 = -t$,

or $\vec{x} = (-3s, s-t, s, -t, t) = s(-3,1,1,0,0) + t(0,-1,0,-1,1)$.

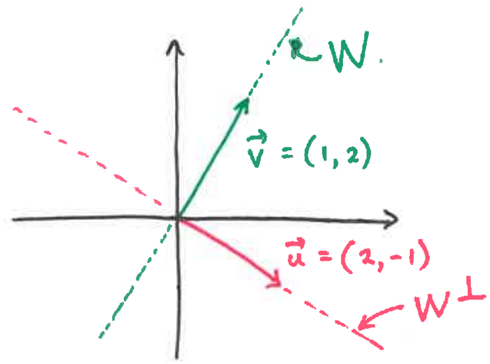
basis for null$(A) = \{(-3,1,1,0,0), (0,-1,0,-1,1)\}$.

(e) What is the nullity of $A$?

nullity$(A) = 2$ because ... dim(null$(A)$) = 2 or $A$ has 2 free variables or $n - \text{rank}(A) = 5 - 3 = \underline{2}$.

If $W$ is a subspace of $\mathbb{R}^n$, then the set of all vectors in $\mathbb{R}^n$ that are orthogonal to *every* vector in $W$ is called the orthogonal complement of $W$, and is denoted by $W^\perp$. ← "W perp"

**Example 3.** Let $W = \text{span}\{(1,2)\}$, which is a subspace of $\mathbb{R}^2$.




(a) Find a vector in $W^\perp$. $\vec{u} = (2,-1)$.

(why? if $\vec{v}$ is in $W$, then $\vec{v} = k(1,2) = (k,2k)$. Thus $(2,-1)\cdot(k,2k) = 2k - 2k = 0$.)

(b) Describe the set of all vectors in $W^\perp$.

$W^\perp$ contains all vector parallel to $(2,-1)$.

(why? $(2\ell,-\ell)\cdot(k,2k) = 2k\ell - 2k\ell = 0$.)

note: $\{\vec{0}\}$ is the orthogonal complement of $\mathbb{R}^2$ in $\mathbb{R}^2$.

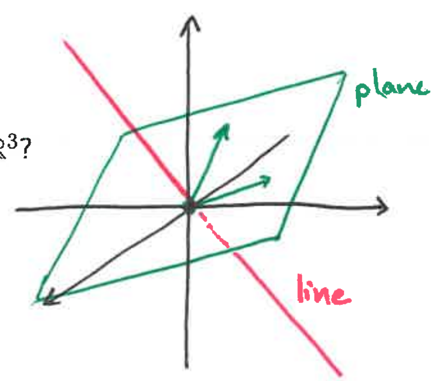
**Theorem.** If $W$ is a subspace of $\mathbb{R}^n$, then:

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.
2. The only vector in both $W$ and $W^\perp$ is $\vec{0}$.
3. The orthogonal complement of $W^\perp$ is $W$.

**Example 4.** (a) What is the orthogonal complement of a line through the origin in $\mathbb{R}^3$?

a plane through the origin.

(i.e. the plane that is orthogonal to any vector on the line)




(b) What is the orthogonal complement of a plane through the origin in $\mathbb{R}^3$?

a line through the origin.

(i.e. the line that is orthogonal to any vector on the plane)

Recall that if $\vec{x}_h$ is a solution to the homogeneous linear system $A\vec{x} = \vec{0}$, then $\vec{x}_h$ is orthogonal to every row of $A$. That is, $\vec{x}_h \cdot \vec{r}_i = 0$ where $\vec{r}_i$ is the $i$th row of $A$.

**Theorem.** If $A$ is an $m \times n$ matrix, then:

1. The null space of $A$ and the row space of $A$ are orthogonal complements in $\mathbb{R}^n$.
2. The null space of $A^T$ and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$.

**Example 5.** Let $\vec{x}_h$ be a solution to the homogeneous linear system $A\vec{x} = \vec{0}$, and let $\vec{r}$ be a vector in the row space of $A$. Show that $\vec{x}_h$ is orthogonal to $\vec{r}$.

Because $\vec{r}$ is in the row space of $A$, we can write

$$\vec{r} = c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_m\vec{r}_m$$

where $\vec{r}_i$ is the $i^{th}$ row of $A$.

Then:

$$\begin{aligned}
\vec{x}_h \cdot \vec{r} &= \vec{x}_h \cdot (c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_m\vec{r}_m) \\
&= \vec{x}_h \cdot (c_1\vec{r}_1) + \vec{x}_h \cdot (c_2\vec{r}_2) + \cdots + \vec{x}_h \cdot (c_m\vec{r}_m) \\
&= c_1\, \vec{x}_h \cdot \vec{r}_1 + c_2\, \vec{x}_h \cdot \vec{r}_2 + \cdots + c_m\, \vec{x}_h \cdot \vec{r}_m \\
&= c_1(0) + c_2(0) + \cdots + c_m(0) \\
&= 0.
\end{aligned}$$

That is, $\vec{x}_h$ is orthogonal to $\vec{r}$.

<u>note:</u> This proves part (1) of the theorem above, because we have shown that any vector in null$(A)$ is orthogonal to any vector in row$(A)$.

We finally have all the ingredients to state the "Equivalence Theorem" in full.

**Equivalence Theorem.** If $A$ is an $n \times n$ matrix with no repeated rows or repeated columns, then the following statements are equivalent.

1. $A$ is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of $A$ is $I_n$.
4. $A$ can be written as a product of elementary matrices.
5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.
7. $\det A \neq 0$.
8. The column vectors of $A$ are linearly independent.
9. The row vectors of $A$ are linearly independent.
10. The column vectors of $A$ span $\mathbb{R}^n$.
11. The row vectors of $A$ span $\mathbb{R}^n$.
12. The column vectors of $A$ are a basis for $\mathbb{R}^n$.
13. The row vectors of $A$ are a basis for $\mathbb{R}^n$.
14. $\text{rank}(A) = n$.
15. $\text{nullity}(A) = 0$.
16. The orthogonal complement of $\text{null}(A)$ is $\mathbb{R}^n$.
17. The orthogonal complement of $\text{row}(A)$ is $\{\vec{0}\}$.

## Section 5.1 Eigenvalues and Eigenvectors

"eigen" = "own"

**Objectives.**

- Introduce eigenvalues and eigenvectors for a matrix or matrix transformation.
- Find eigenvalues, eigenvectors, and eigenspaces.

---

Suppose that $\vec{x}$ is a non-zero vector and $A$ is a square matrix. If $A\vec{x} = \lambda\vec{x}$ for some scalar $\lambda$, then $\lambda$ is an eigenvalue of $A$ and $\vec{x}$ is an eigenvector of $A$ corresponding to $\lambda$.

**Example 1.** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Compute $A\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Which of these vectors is an eigenvector of $A$? What are the eigenvalues?

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ i.e. $\lambda = 1$.

Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $A$ with eigenvalue $\lambda = 1$.

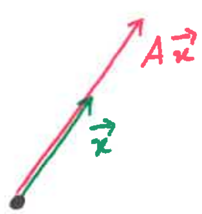
$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ i.e. $\lambda = 2$.

Thus $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ with eigenvalue $\lambda = 2$.

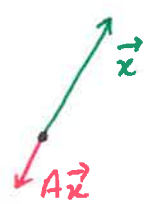
$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \lambda\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not an eigenvector of $A$.

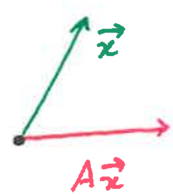
Loosely speaking, an eigenvector of an $n \times n$ matrix $A$ (or of the matrix operator $T_A$) is a direction in $\mathbb{R}^n$ that is unchanged when multiplying by $A$. That is, $\vec{x} \neq \vec{0}$ is an eigenvector of $A$ if $\vec{x}$ and $A\vec{x}$ are parallel.




$\vec{x}$ is an eigenvector of $A$ with $\lambda > 1$.




$\vec{x}$ is an eigenvector of $A$ with $-1 < \lambda < 0$.




$\vec{x}$ is not an eigenvector of $A$.

(b/c $A\vec{x} \neq \lambda\vec{x}$)

characteristic equation of $A$.



**Theorem.** If $A$ is a square matrix, then $\lambda$ is an eigenvalue of $A$ if and only if $\det(\lambda I - A) = 0$.

**Proof.** Suppose $\lambda$ is an eigenvalue of $A$. Then there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. That is, $A\vec{x} = \lambda I \vec{x}$, so $\vec{0} = \lambda I\vec{x} - A\vec{x} = (\lambda I - A)\vec{x}$. Thus $\det(\lambda I - A) = 0$.

Suppose $\det(\lambda I - A) = 0$. Then there is a nonzero vector $\vec{x}$ such that $(\lambda I - A)\vec{x} = \vec{0}$. Thus $\lambda I\vec{x} - A\vec{x} = \vec{0}$, so

$A\vec{x} = \lambda I \vec{x} = \lambda \vec{x}$. Therefore, $\lambda$ is an eigenvalue of $A$.

**Example 2.** Use the theorem above to find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{bmatrix}.$$

characteristic polynomial of $A$.

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{bmatrix} = (\lambda-1)(\lambda-2) - (-1)(0) = (\lambda-1)(\lambda-2).$$

- solve $\det(\lambda I - A) = 0$:

  $(\lambda-1)(\lambda-2) = 0 \implies \lambda = 1, 2.$

The eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 2$.

**Strategy.** To find the eigenvalues of $A$:

- set up the characteristic equation/polynomial of $A$
- find all the solutions of the characteristic equations.

**Example 3.** Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$.

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-1 & -2 & 0 \\ -3 & \lambda-1 & -2 \\ 0 & -3 & \lambda-1 \end{bmatrix} = (\lambda-1)\left((\lambda-1)^2-6\right) - (-2)\left((-3)(\lambda-1)\right)$$

$$= (\lambda-1)\left((\lambda-1)^2-6-6\right) = \underbrace{(\lambda-1)(\lambda^2-2\lambda-11)}_{\text{characteristic polynomial}}.$$

$\det(\lambda I - A) = 0 \implies \lambda = 1$ or $\lambda^2 - 2\lambda - 11 = 0 \leftarrow$ use quadratic formula!!!

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4+44}}{2} = 1 \pm \frac{\sqrt{48}}{2} = 1 \pm 2\sqrt{3}.$$

The eigenvalues are $\lambda = 1$, $\lambda = 1 + 2\sqrt{3}$, $\lambda = 1 - 2\sqrt{3}$.

The eigenvalues of a triangular matrix can be found 'by inspection' (that is, without solving the characteristic polynomial).

| |
|---|
| **Theorem.** If $A$ is triangular, then the eigenvalues of $A$ are the entries on the main diagonal. |

**Example 4.** Find the eigenvalues of each matrix.

$$\begin{bmatrix} 3 & 9 & -4 \\ 0 & -7 & 5 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 1 & -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

$\lambda = 3, -7, 4$. $\qquad \lambda = \frac{1}{2}, \frac{3}{2}, 2$. $\qquad \lambda = a, b, c, d$.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-3 & -9 & 4 \\ 0 & \lambda+7 & -5 \\ 0 & 0 & \lambda-4 \end{bmatrix}$$

$$= (\lambda-3)(\lambda+7)(\lambda-4).$$

**Theorem.** If $A$ is a square matrix, then the following statements are equivalent.

1. $\lambda$ is an eigenvalue of $A$.
2. $\lambda$ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.
3. The system $(\lambda I - A)\vec{x} = \vec{0}$ has nontrivial solutions.
4. There is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$.

Now that we know how to find eigenvalues for a matrix, we turn our attention to finding the eigenvectors corresponding to each eigenvalue. If $\lambda$ is an eigenvalue of $A$, then the eigenvectors corresponding to $\lambda$ are the nonzero vectors $\vec{x}$ such that $(\lambda I - A)\vec{x} = \vec{0}$. This solution space is the eigenspace corresponding to $\lambda$.

- find all eigenvalues of $A$
- solve $(\lambda I - A)\vec{x} = \vec{0}$ for each eigenvalue $\lambda$.

**Example 5.** Find the eigenspaces of the matrix $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$.

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix} = (\lambda+1)\lambda - 6 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2).$$

The eigenvalues of $A$ are $\lambda = -3$, $\lambda = 2$.

$\lambda = 2$:

$$\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{use "elimination" to solve.}}$$

$\Longrightarrow x_1 = t, \; x_2 = t.$

Thus $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to $\lambda = 2$.

$\lambda = -3$:

$$\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow x_1 = -\frac{3}{2}t, \; x_2 = t.$

Thus $\left\{\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to $\lambda = -3$.

cofactor expansion!!!

**Example 6.** Find the eigenspaces of the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} = \cdots = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda-1)(\lambda-2)^2.$$

The eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 2$. ← repeated eigenvalue.

$\underline{\lambda = 1:}$ $\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -2s, \ x_2 = s, \ x_3 = s.$

The ~~eigenvalue~~ eigenvectors for $\lambda = 1$ are $s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, so $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace.

$\underline{\lambda = 2:}$ $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -t, \ x_2 = s, \ x_3 = t$

The eigenvectors for $\lambda = 2$ are $\begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, so $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace.

**Theorem.** The square matrix $A$ is invertible if and only if $\lambda = 0$ is not an eigenvalue of $A$.

**Example 7.** Find the eigenvalues of $A = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$.

$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-1 & 3 \\ 0 & \lambda \end{bmatrix} = (\lambda-1)\lambda$, so $\lambda = 0, 1$ are the eigenvalues of $A$. Thus $A$ is $\underline{\text{not}}$ invertible.

## Section 5.2 Diagonalization

**Objectives.**

- Define similarity transformations and identify some properties of similar matrices.
- Introduce the idea of diagonalizing a matrix.
- Use diagonalization to compute powers of a matrix efficiently.

---

Let $A$ and $P$ be $n \times n$ matrices with $P$ invertible. The transformation that sends $A$ to the matrix product $P^{-1}AP$ is called a similarity transformation.

More generally, if $A$ and $B$ are $n \times n$ matrices then we say that $B$ is similar to $A$ if there is an invertible matrix $P$ such that $B = P^{-1}AP$.

**Example 1.** Suppose that $B$ is similar to $A$. Show that $A$ is similar to $B$.

Because $B$ is similar to $A$, there is an invertible matrix $P$ such that $B = P^{-1}AP$.

Then $PBP^{-1} = P(P^{-1}AP)P^{-1} = (PP^{-1})A(PP^{-1}) = IAI = A$.

That is $A = Q^{-1}BQ$ where $Q = P^{-1}$. Thus $A$ is similar to $B$.

(Notice that the previous example allows us to say that $A$ and $B$ are similar if one is similar to the other.)

Similar matrices share several important properties. In particular, if $A$ and $B$ are similar then $A$ and $B$ have the same ...

determinant, rank, nullity, trace, characteristic polynomial, eigenvalues, ...

note: similar matrices represent the same linear transformation with respect to different bases.

**Example 2.** Suppose that $A$ and $B$ are similar matrices. Show that $\det(A) = \det(B)$.

Because $A$ and $B$ are similar, there is an invertible matrix $P$ such that $B = P^{-1}AP$.

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$
$$= \frac{1}{\det(P)}\det(A)\det(P) = \det(A).$$

An $n \times n$ matrix $A$ is diagonalizable if it is similar to a diagonal matrix. That is, if there is an invertible matrix $P$ such that $P^{-1}AP$ is diagonal, in which case we say that $P$ diagonalizes $A$.

**Example 3.** Consider the $2 \times 2$ matrices $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Show that $P$ diagonalizes $A$.

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2-1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}.$$

Thus $P$ ~~diagonal~~ diagonalizes $A$.

(b) What are the eigenvalues of $A$?

$P^{-1}AP$ has eigenvalues $\lambda = 4, 5$, so $A$ also has eigenvalues $\lambda = 4, 5$.

The key ingredient for diagonalizing a matrix is the set of eigenvectors of the matrix.

**Theorem.** An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

**Theorem.** If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix $A$, and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are corresponding eigenvectors, then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

It follows from the previous two theorems that an $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

Why? each eigenvalue corresponds to (at least) one eigenvector, so $n$ distinct eigenvalues gives us $n$ linearly independent eigenvectors.

Thus these $n$ eigenvectors are a basis for $\mathbb{R}^n$.

**Strategy.** To find a matrix that diagonalizes $A$:

- find the eigenvalues and corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_k$ of $A$.
- if you find $n$ eigenvectors, then $P = [\vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_n]$ diagonalizes $A$.

**Example 4.** Find a matrix $P$ that diagonalizes $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

From Ex. 6, Section 5.1, the eigenvalues of $A$ are $\lambda = 1$ (with eigenvector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$) and $\lambda = 2$ (with eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$). These three eigenvectors are linearly independent, so $P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalizes $A$.

eigenvalues of $A$!!

check: $$P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

note: if we chose $P = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, then $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**Example 5.** Show that the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ is not diagonalizable.

$A$ is triangular, so the eigenvalues are $\lambda = 1$ (repeated) and $\lambda = 4$.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 4 \end{bmatrix} = (\lambda - 1)^2(\lambda - 4). \Rightarrow \lambda = 1, \lambda = 4.$$

$\lambda = 1$: $$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = t, \ x_2 = 0, \ x_3 = 0.$$

The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for this eigenspace.

$\lambda = 4$: $$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = t, \ x_2 = 3t, \ x_3 = 9t.$$

The set $\left\{ \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \right\}$ is a basis for the eigenspace.

Because $A$ has only <u>two</u> linearly independent eigenvectors, we cannot diagonalize $A$.

**Example 6.** Explain why the matrix $A = \begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ is diagonalizable.

The eigenvalues of $A$ are $\lambda = 1, 2, 4, 5$. These are distinct, so $A$ has four linearly independent eigenvectors. Therefore $A$ is diagonalizable.

One application of diagonalization is finding powers of a matrix. Recall that if $D$ is a diagonal matrix, then $D^k$ can be found by raising each diagonal entry to the power $k$.

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix} \quad (\text{for } k > 0).$$

Suppose that $A$ is similar to a diagonal matrix $D$, so that $A = P^{-1}DP$ where $P$ is invertible. Then:

i.e. $A$ is diagonalizable.

$$A^k = (P^{-1}DP)^k = (P^{-1}DP)(P^{-1}DP)\cdots(P^{-1}DP)$$
$$= P^{-1}D(PP^{-1})D(PP^{-1})D\cdots DP = P^{-1}D^kP.$$

**Example 7.** Compute $A^5$ for the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ in Example 4.

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Thus: $$A^5 = (PDP^{-1})^5 = PD^5P^{-1} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$
$$= \cdots = \begin{bmatrix} -30 & 0 & -62 \\ 31 & 32 & 31 \\ 31 & 0 & 63 \end{bmatrix}.$$