Section 5.2 Diagonalization

Objectives.

- Define similarity transformations and identify some properties of similar matrices.
- Introduce the idea of diagonalizing a matrix.
- Use diagonalization to compute powers of a matrix efficiently.

Let A and P be $n \times n$ matrices with P invertible. The transformation that sends A to the matrix product $P^{-1}AP$ is called a similarity transformation.

More generally, if A and B are $n \times n$ matrices then we say that B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$.

Example 1. Suppose that B is similar to A. Show that A is similar to B.

(Notice that the previous example allows us to say that A and B are similar if one is similar to the other.)

Similar matrices share several important properties. In particular, if A and B are similar then A and B have the same ...

Example 2. Suppose that A and B are similar matrices. Show that $\det(A) = \det(B)$.

An $n \times n$ matrix A is diagonalizable if it is similar to a diagonal matrix. That is, if there is an invertible matrix P such that $P^{-1}AP$ is diagonal, in which case we say that P diagonalizes A.

Example 3. Consider the 2×2 matrices $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Show that P diagonalizes A.

(b) What are the eigenvalues of A?

The key ingredient for diagonalizing a matrix is the set of eigenvectors of the matrix.

Theorem. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Theorem. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix A, and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are corresponding eigenvectors, then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

It follows from the previous two theorems that an $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Strategy. To find a matrix that diagonalizes A:

Example 4. Find a matrix P that diagonalizes $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Example 5. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ is not diagonalizable.

Example 6. Explain why the matrix $A = \begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ is diagonalizable.

One application of diagonalization is finding powers of a matrix. Recall that if D is a diagonal matrix, then D^k can be found by raising each diagonal entry to the power k.

Suppose that A is similar to a diagonal matrix D, so that $A = P^{-1}DP$ where P is invertible. Then:

Example 7. Compute A^5 for the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ in Example 4.