Section 5.1 Eigenvalues and Eigenvectors

Objectives.

- Introduce eigenvalues and eigenvectors for a matrix or matrix transformation.
- Find eigenvalues, eigenvectors, and eigenspaces.

Suppose that $\vec{x}$ is a non-zero vector and A is a square matrix. If $A\vec{x} = \lambda\vec{x}$ for some scalar λ, then λ is an eigenvalue of A and $\vec{x}$ is an eigenvector of A corresponding to λ.

Example 1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Compute $A\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Which of these vectors is an eigenvector of A? What are the eigenvalues?

Loosely speaking, an eigenvector of an $n \times n$ matrix A (or of the matrix operator T_A) is a direction in $\mathbb{R}^n$ that is unchanged when multiplying by A. That is, $\vec{x} \neq \vec{0}$ is an eigenvector of A if $\vec{x}$ and $A\vec{x}$ are parallel.

Theorem. If A is a square matrix, then λ is an eigenvalue of A if and only if $\det(\lambda I - A) = 0$.

Proof.

Example 2. Use the theorem above to find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Strategy. To find the eigenvalues of A:

Example 3. Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$.

The eigenvalues of a triangular matrix can be found 'by inspection' (that is, without solving the characteristic polynomial).

Theorem. If A is triangular, then the eigenvalues of A are the entries on the main diagonal.

Example 4. Find the eigenvalues of each matrix.

$$\begin{bmatrix} 3 & 9 & -4 \\ 0 & -7 & 5 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 1 & -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

Theorem. If A is a square matrix, then the following statements are equivalent.

1. λ is an eigenvalue of A.
2. λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.
3. The system $(\lambda I - A)\vec{x} = \vec{0}$ has nontrivial solutions.
4. There is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$.

Now that we know how to find eigenvalues for a matrix, we turn our attention to finding the eigenvectors corresponding to each eigenvalue. If λ is an eigenvalue of A, then the eigenvectors corresponding to λ are the nonzero vectors $\vec{x}$ such that $(\lambda I - A)\vec{x} = \vec{0}$. This solution space is the eigenspace corresponding to λ.

Example 5. Find the eigenspaces of the matrix $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$.

Example 6. Find the eigenspaces of the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Theorem. The square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Example 7. Find the eigenvalues of $A = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$.