Section 4.9 Rank, Nullity, and the Fundamental Matrix Spaces

Objectives.

- Define the rank and nullity of a matrix, and see how these are related.
- Introduce the orthogonal complement of a subspace.
- Extend the Equivalence Theorem.

Recall the following definitions from Section 4.8.

- the row space of A is the set of all linear combinations of the row vectors of A
- the column space of A is the set of all linear combinations of the column vectors of A
- the null space of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$

The dimensions of these three spaces are related, and depend on the number of "leading variables" and "free variables" in a linear system.

Theorem. The row space and column space of a matrix A have the same dimension.

The common dimension of the row space and the column space of A is called the rank of A. The dimension of the null space of a matrix A is called the nullity of A.

Example 1. What is the rank of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$? What is the nullity of A?

Theorem. If A is an $m \times n$ matrix, then $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$.

We can also relate the rank and nullity of a matrix with the number of leading variables and the number of free variables in a homogeneous linear system.

Theorem. Let A be an $m \times n$ matrix. Then $\operatorname{rank}(A)$ is the number of leading variables in the general solution to $A\vec{x} = \vec{0}$, and $\operatorname{nullity}(A)$ is the number of free variables in the general solution to $A\vec{x} = \vec{0}$.

Example 2. The matrices A, B, and C below are row equivalent.

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 1 & 2 & 1 & 0 & 2 \\ 2 & 4 & 2 & 1 & 5 \\ 1 & 0 & 3 & -2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for $\operatorname{row}(A)$.

(b) Find a basis for $\operatorname{col}(A)$.

(c) What is the rank of A?

(d) Find a basis for $\operatorname{null}(A)$.

(e) What is the nullity of A?

If W is a subspace of $\mathbb{R}^n$, then the set of all vectors in $\mathbb{R}^n$ that are orthogonal to *every* vector in W is called the orthogonal complement of W, and is denoted by $W^\perp$.

Example 3. Let $W = \operatorname{span}\{(1,2)\}$, which is a subspace of $\mathbb{R}^2$.

(a) Find a vector in $W^\perp$.

(b) Describe the set of all vectors in $W^\perp$.

Theorem. If W is a subspace of $\mathbb{R}^n$, then:

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.
2. The only vector in both W and $W^\perp$ is $\vec{0}$.
3. The orthogonal complement of $W^\perp$ is W.

Example 4. (a) What is the orthogonal complement of a line through the origin in $\mathbb{R}^3$?

(b) What is the orthogonal complement of a plane through the origin in $\mathbb{R}^3$?

Recall that if $\vec{x}_h$ is a solution to the homogeneous linear system $A\vec{x} = \vec{0}$, then $\vec{x}_h$ is orthogonal to every row of A. That is, $\vec{x}_h \cdot \vec{r}_i = 0$ where $\vec{r}_i$ is the ith row of A.

Theorem. If A is an $m \times n$ matrix, then:

1. The null space of A and the row space of A are orthogonal complements in $\mathbb{R}^n$.
2. The null space of A^T and the column space of A are orthogonal complements in $\mathbb{R}^m$.

Example 5. Let $\vec{x}_h$ be a solution to the homogeneous linear system $A\vec{x} = \vec{0}$, and let $\vec{r}$ be a vector in the row space of A. Show that $\vec{x}_h$ is orthogonal to $\vec{r}$.

We finally have all the ingredients to state the "Equivalence Theorem" in full.

Equivalence Theorem. If A is an $n \times n$ matrix with no repeated rows or repeated columns, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.
5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.
7. $\det A \neq 0$.
8. The column vectors of A are linearly independent.
9. The row vectors of A are linearly independent.
10. The column vectors of A span $\mathbb{R}^n$.
11. The row vectors of A span $\mathbb{R}^n$.
12. The column vectors of A are a basis for $\mathbb{R}^n$.
13. The row vectors of A are a basis for $\mathbb{R}^n$.
14. $\text{rank}(A) = n$.
15. $\text{nullity}(A) = 0$.
16. The orthogonal complement of $\text{null}(A)$ is $\mathbb{R}^n$.
17. The orthogonal complement of $\text{row}(A)$ is $\{\vec{0}\}$.