Section 4.7 Change of Basis

Objectives.

- Introduce the 'change of basis problem'.
- Define the transition matrix for a change of basis.
- Find the transition matrix for a change of basis.

Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for a vector space V, and let $\vec{v}$ be a vector in V. Recall the definition of the coordinate vector for $\vec{v}$ relative to B:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \iff [\vec{v}]_B = (c_1, c_2, \cdots, c_n) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

The set of all such coordinate vectors for V is a function from V to $\mathbb{R}^n$ called the coordinate map relative to B.

i.e. for each vector $\vec{v}$ in V and each basis B for V, there is a coordinate vector $[\vec{v}]_B$ in $\mathbb{R}^n$.

Sometimes we may want to change from one basis B for V to a different basis B'. Thus we would like to know how $[\vec{v}]_B$ and $[\vec{v}]_{B'}$ are related.

Suppose that $B = \{\vec{u}_1, \vec{u}_2\}$ and $B' = \{\vec{u}_1', \vec{u}_2'\}$ are both bases for V, and that $\vec{v}$ is a vector in V.

Let $[\vec{u}_1]_{B'} = \begin{bmatrix} a \\ b \end{bmatrix}$, $[\vec{u}_2]_{B'} = \begin{bmatrix} c \\ d \end{bmatrix}$, and $[\vec{v}]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

Then: $\vec{u}_1 = a\vec{u}_1' + b\vec{u}_2'$ and $\vec{u}_2 = c\vec{u}_1' + d\vec{u}_2'$, so

$$\vec{v} = k_1\vec{u}_1 + k_2\vec{u}_2 = k_1(a\vec{u}_1' + b\vec{u}_2') + k_2(c\vec{u}_1' + d\vec{u}_2') = (k_1a + k_2c)\vec{u}_1' + (k_1b + k_2d)\vec{u}_2'$$

Thus: $$[\vec{v}]_{B'} = \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}[\vec{v}]_B.$$

transition matrix from B to B'

$$P_{B\to B'} = \Big[[\vec{u}_1]_{B'} \;\; [\vec{u}_2]_{B'} \Big].$$

Change of Basis Problem. Suppose that $B = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is the old basis for V, and $B' = \{\vec{u}_1', \vec{u}_2', \dots, \vec{u}_n'\}$ is the new basis for V. Then the coordinate vectors for a vector $\vec{v}$ in V satisfy

$$[\vec{v}]_{B'} = P_{B\to B'}\,[\vec{v}]_B$$

where $P_{B\to B'} = \big[[\vec{u}_1]_{B'} \quad [\vec{u}_2]_{B'} \quad \cdots \quad [\vec{u}_n]_{B'}\big]$ is the transition matrix from B to B'.

The columns of the transition matrix are *the coordinate vectors of the old basis relative to the new basis.*

Example 1. Consider the bases $B = \{(1,0),(0,1)\}$ and $B' = \{(1,1),(1,2)\}$ for $\mathbb{R}^2$.

(a) Find the transition matrix $P_{B\to B'}$ from B to B'.

$\vec{u}_1 = (1,0) = 2(1,1) - (1,2) = 2\vec{u}_1' - \vec{u}_2'$, so $[\vec{u}_1]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$\vec{u}_2 = (0,1) = -(1,1) + (1,2) = -\vec{u}_1' + \vec{u}_2'$, so $[\vec{u}_2]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Thus: $P_{B\to B'} = \big[[\vec{u}_1]_{B'} \; [\vec{u}_2]_{B'}\big] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

(b) Find the transition matrix $P_{B'\to B}$ from B' to B.

$\vec{u}_1' = (1,1) = (1,0) + (0,1) = \vec{u}_1 + \vec{u}_2$, so $[\vec{u}_1']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\vec{u}_2' = (1,2) = \cdots = \vec{u}_1 + 2\vec{u}_2$, so $[\vec{u}_2']_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Thus: $P_{B'\to B} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(c) Suppose that $[\vec{v}]_B = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Find $[\vec{v}]_{B'}$.

$[\vec{v}]_{B'} = P_{B\to B'}[\vec{v}]_B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}$.

Applying a change of basis from B to B' and then a change of basis from B' to B leaves coordinate vectors unchanged.

i.e.
$$[\vec{v}]_B = P_{B'\to B}[\vec{v}]_{B'} = P_{B'\to B}\left(P_{B\to B'}[\vec{v}]_B\right)$$
$$= \left(P_{B'\to B}P_{B\to B'}\right)[\vec{v}]_B = I[\vec{v}]_B,$$

so
$$P_{B'\to B}P_{B\to B'} = I.$$

This means that the transition matrices $P_{B\to B'}$ and $P_{B'\to B}$ are inverses of each other.

Theorem. If P is the transition matrix from a basis B to a basis B' in the vector space V, then P is invertible and P^{-1} is the transition matrix from B' to B.

We can find a transition matrix by row-reducing the matrix that has the vectors from each basis as columns.

$$\left[\begin{array}{cc|cc} \vec{u}_1' & \vec{u}_2' & \vec{u}_1 & \vec{u}_2 \end{array}\right] \xrightarrow{\text{row operations}} \left[\begin{array}{c|c} I & P_{B\to B'} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} \vec{u}_1' & \vec{u}_2' & \vec{u}_1 & \vec{u}_2 \end{array}\right] \xrightarrow{\text{row operations}} \left[\begin{array}{c|c} P_{B'\to B} & I \end{array}\right]$$

new basis B' — old basis B

Example 2. Find the transition matrix $P_{B\to B'}$ for the bases in Example 1.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - R_1]{} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 - R_2]{} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Thus
$$P_{B\to B'} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Theorem. Let $B = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be any basis for $\mathbb{R}^n$ and let $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then the transition matrix from B to S is
$$P_{B\to S} = [\vec{u}_1|\vec{u}_2|\cdots|\vec{u}_n].$$

In particular, if $A = [\vec{v}_1|\vec{v}_2|\cdots|\vec{v}_n]$ is an invertible matrix, then A is a transition matrix from the basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ to the standard basis for $\mathbb{R}^n$.