Section 4.6 Dimension

Objectives.

- Define the dimension of a finite-dimensional vector space.
- Relate dimension to span and linear independence.

Theorem. Every basis for a finite-dimensional vector space V contains the same number of vectors.

The number of vectors in a basis for the finite-dimensional vector space V is called the dimension of V, and is denoted by $\dim V$.

note: if $V = \{\vec{0}\}$, then $\dim V = 0$.

Example 1. What is the dimension of each vector space?

(a) $\mathbb{R}^n$ The standard basis is $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$.

Thus $\dim(\mathbb{R}^n) = n$.

(b) P_n The standard basis is $\{1, x, \dots, x^n\}$.

Thus $\dim(P_n) = n+1$.

(c) M_{mn}

$\dim(M_{mn}) = mn$. e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in the standard basis for M_{23}.

Theorem. Let V be a finite-dimensional vector space with $\dim V = n$.

1. If W is a subset of V that contains more than n vectors, then W is linearly dependent.
2. If W is a subset of V that contains fewer than n vectors, then W does not span V.

Example 2. Suppose that $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ is a linearly independent set of vectors in a vector space V. What is $\dim(\text{span}(S))$? Why?

S is linearly independent, and S spans $\text{span}(S)$. This means that S is a basis for $\text{span}(S)$, so $\dim(\text{span}(S)) = r$.

Example 3. Consider the linear system below. (This is Example 5 from the Section 1.2 lecture notes.)

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

The general solution to this system is

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0.$$

(a) Write the solution in vector form.

$$\begin{aligned} \vec{x} &= (-3r-4s-2t, r, -2s, s, t, 0) \\ &= r(-3,1,0,0,0,0) + s(-4,0,-2,1,0,0) + t(-2,0,0,0,1,0). \end{aligned}$$

these vectors are linearly independent!!!

(b) Find a basis for the solution space of the system.

Every vector $\vec{x}$ in the solution space is a linear combination of $(-3,1,0,0,0,0)$, $(-4,0,-2,1,0,0)$, $(-2,0,0,0,1,0)$, and these vectors are linearly independent. Thus $\{(-3,1,0,0,0,0), (-4,0,-2,1,0,0), (-2,0,0,0,1,0)\}$ is a basis for the solution space.

(c) What is the dimension of the solution space?

There are three vectors in <u>any</u> basis, so this space has dimension 3.

"union" (i.e. add $\vec{v}$ to the set S).

Theorem. Let S be a set of vectors in a vector space V.

1. If S is linearly independent, and $\vec{v}$ is not in $\mathrm{span}(S)$, then $S \cup \{\vec{v}\}$ is linearly independent.
   i.e. adding a vector outside $\mathrm{span}(S)$ does not affect linear independence.
2. If $\vec{v}$ is in S, and $\vec{v}$ can be written as a (nonzero) linear combination of other vectors in S, then

remove $\vec{v}$ from S.

$$\mathrm{span}(S) = \mathrm{span}(S - \{\vec{v}\}).$$

i.e. removing linearly dependent vectors does not affect $\mathrm{span}(S)$.

Example 4. Explain why the polynomials $p(x) = 1 + x^2$, $q(x) = 2 + x^2$, $r(x) = x^3$ are linearly independent.

$p(x)$ and $q(x)$ are linearly independent (neither is a multiple of the other). Also, $r(x)$ is not in $\mathrm{span}\{p(x), q(x)\}$, because r is cubic but p, q are quadratic. Thus $\{p(x), q(x), r(x)\}$ is linearly independent.

Theorem. Let V be a vector space with $\dim V = n$, and let S be a set of n vectors in V.

equal!!!

1. S is a basis for V if and only if S is linearly independent.
2. S is a basis for V if and only if S spans V.

Example 5. Explain why each set of vectors is a basis for the given vector space.

(a) $\vec{v}_1 = (1, 4)$ and $\vec{v}_2 = (3, -2)$ in $\mathbb{R}^2$

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent, and $\dim(\mathbb{R}^2) = 2$.

Thus $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$.

(b) $\vec{v}_1 = (1, 0, 2)$, $\vec{v}_2 = (-1, 0, 1)$, and $\vec{v}_3 = (2, -2, 3)$ in $\mathbb{R}^3$

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent in the xz-plane.

because the y-coord. is $\neq 0$.

Because $\vec{v}_3$ is not in the xz-plane (i.e. $\vec{v}_3$ is not in $\mathrm{span}\{\vec{v}_1, \vec{v}_2\}$), the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Also, $\dim(\mathbb{R}^3) = 3$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Theorem. Let V be a vector space with $\dim V = n$, and let S be a set of vectors in V.

1. If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing some vectors.
2. If S is linearly independent but is not a basis for V, then S can be enlarged to a basis for V by adding some vectors.

Example 6. (a) Find a subset of $S = \{(1,-1),(-1,1),(1,1)\}$ that is a basis for $\mathbb{R}^2$.

$\dim(\mathbb{R}^2) = 2$, so we need two vectors from S to form a basis.

The vectors $(1,-1)$ and $(1,1)$ are linearly independent.

Thus $\{(1,-1),(1,1)\}$ is a basis for $\mathbb{R}^2$.

note: $\{(-1,1),(1,1)\}$ is also a basis for $\mathbb{R}^2$, but

$\{(1,-1),(-1,1)\}$ is not a basis for $\mathbb{R}^2$. (why?)

(b) Enlarge the set $S = \{(1,1,0),(1,0,-1)\}$ to a basis for $\mathbb{R}^3$.

Let's try adding $(1,0,0)$ to S.

$k_1(1,1,0) + k_2(1,0,-1) + k_3(1,0,0) = (0,0,0)$

$\Rightarrow (k_1 + k_2 + k_3, k_1, -k_2) = (0,0,0)$

$\Rightarrow k_1 = k_2 = k_3 = 0$

Thus $\{(1,1,0),(1,0,-1),(1,0,0)\}$ is linearly independent and contains three vectors, so this is a basis for $\mathbb{R}^3$.

note: $\{(1,1,0),(1,0,-1),(0,1,0)\}$ and $\{(1,1,0),(1,0,-1),(0,0,1)\}$ are also bases for $\mathbb{R}^3$.

Theorem. If W is a subspace of a finite-dimensional vector space V, then:

1. W is finite-dimensional.
2. $\dim W \leq \dim V$.
3. $W = V$ if and only if $\dim W = \dim V$.