Section 4.5 Coordinates and Basis

Objectives.

- Introduce the idea of a basis for a vector space.
- Find coordinates for a vector relative to a given basis.

A vector space V is finite-dimensional if there is a finite set of vectors S that spans V. Otherwise, V is infinite-dimensional.

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors in a finite-dimensional vector space V. We say that S is a basis for V if the following two conditions hold.

- S spans V i.e. every vector in V is a linear combination of vectors in S.
- S is linearly independent i.e. if $k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n = \vec{0}$ then $k_1 = k_2 = \cdots = k_n = 0$.

Example 1. The set $\boxed{S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}}$ is a basis for $\mathbb{R}^n$. "standard basis for $\mathbb{R}^n$"

From Example 1, Section 4.3, $\mathbb{R}^n = \text{span}(S)$.

From Example 1, Section 4.4, S is linearly independent.

Therefore, S is a basis for $\mathbb{R}^n$.

Example 2. The set $\boxed{S = \{1, x, x^2, \ldots, x^n\}}$ is a basis for P_n. "standard basis for P_n"

From Example 3, Section 4.3, $P_n = \text{span}(S)$.

From Example 4, Section 4.4, S is linearly independent.

Therefore, S is a basis for P_n.

Example 3. The vector space P_∞ is infinite-dimensional.

If $S = \{p_1, p_2, \cdots, p_r\}$ is a finite set of polynomials, then ~~there~~ S contains a polynomial of maximum degree, say degree n. Then any linear combination of polynomials in S has degree at most n. Thus we cannot express x^{n+1} as a linear combination of polynomials in S, so S does not span P_∞.

Therefore, P_∞ is infinite-dimensional.

note: $F(-\infty, \infty)$ is also infinite-dimensional.

Example 4. Show that the vectors $\vec{v}_1 = (1,2,1)$, $\vec{v}_2 = (2,9,0)$, $\vec{v}_3 = (3,3,4)$ form a basis for $\mathbb{R}^3$.

- linear independence:

If $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, then

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 \qquad + 4c_3 &= 0. \end{aligned}$$

- spanning set:

If $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = (b_1, b_2, b_3)$, then

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 \qquad + 4c_3 &= b_3. \end{aligned}$$

Because $\det\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} = -1 \neq 0$, the homogeneous system has only the trivial solution (so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent), and the nonhomogeneous is consistent for all vectors (b_1, b_2, b_3) in $\mathbb{R}^3$.

Therefore, $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Example 5. Show that the matrices $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis for the vector space M_{22}.

- linear independence:

If $c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, so M_1, M_2, M_3, M_4 are linearly independent.

- spanning set:

If $c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Taking $c_1 = a$, $c_2 = b$, $c_3 = c$, $c_4 = d$ satisfies this equation, so M_1, M_2, M_3, M_4 span M_{22}.

Therefore, $\boxed{S = \{M_1, M_2, M_3, M_4\}}$ is a basis for M_{22}.

"standard basis for M_{22}"

Theorem. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for the vector space V. Then every vector $\vec{v}$ in V can be written as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

in exactly one way.

Proof. Because S is a basis for V, every vector in V can be written as a linear combination of vectors in S.

Suppose $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ and $\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$.

Then $\vec{0} = (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \cdots + (c_n - d_n)\vec{v}_n$.

Because S is linearly independent, we have $c_1 - d_1 = 0$, $c_2 - d_2 = 0$, $\ldots$, $c_n - d_n = 0$.

Thus $c_1 = d_1$, $c_2 = d_2$, $\ldots$, $c_n = d_n$.

Therefore, $\vec{v}$ ~~be~~ can be written as a linear combination of the basis S in exactly one way.

The numbers $c_1, c_2, \ldots, c_n$ in this theorem are called the coordinates of $\vec{v}$ relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ is called the coordinate vector of $\vec{v}$ relative to the basis S, and is denoted by

$$(\vec{v})_S = (c_1, c_2, \ldots, c_n).$$

If $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$, then

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \iff (\vec{v})_S = (c_1, c_2, \cdots, c_n).$$

Example 6. Consider the standard basis $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for $\mathbb{R}^3$. What is the coordinate vector for $\vec{v} = (a, b, c)$ relative to the basis S?

$\vec{v} = (a, b, c) = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$, so $(\vec{v})_S = (a, b, c)$.

Example 7. Consider the basis $S = \{(1, 0), (1, 2)\}$ for $\mathbb{R}^2$. What is the coordinate vector for $\vec{v} = (-1, 4)$ relative to the basis S?

$\vec{v} = (-1, 4) = -3(1, 0) + 2(1, 2)$, so $(\vec{v})_S = (-3, 2)$.

Example 8. Find the coordinate vector for the polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ relative to the standard basis for P_n.

$$p(x) = a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n),$$

$$\text{so} \quad (p(x))_S = (a_0, a_1, a_2, \dots, a_n).$$

Example 9. Find the coordinate vector for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ relative to the standard basis for M_{22}.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\text{so} \quad (A)_S = (a, b, c, d).$$

Example 10. Recall from Example 4 that $\vec{v}_1 = (1, 2, 1)$, $\vec{v}_2 = (2, 9, 0)$, $\vec{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

(a) Find the coordinate vector for $\vec{v} = (5, -1, 9)$ relative to the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

From $(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$, we obtain

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 5 \\ 2c_1 + 9c_2 + 3c_3 = -1 \\ c_1 + 4c_3 = 9 \end{cases}.$$

The solution is $c_1 = 1$, $c_2 = -1$, $c_3 = 2$.

Therefore, $(\vec{v})_S = (1, -1, 2)$.

(b) Find $\vec{w}$ given that $(\vec{w})_S = (-1, 3, 2)$.

$$\vec{w} = -1\vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_3 = -1(1,2,1) + 3(2,9,0) + 2(3,3,4)$$

$$= \underline{(11, 31, 7)}.$$