Section 4.4 Linear Independence

Objectives.

- Define linear independence of vectors.
- Determine whether a set of vectors is linearly independent or linearly dependent.
- Define and apply the Wronskian to determine whether a set of functions is linearly independent.

Let V be a vector space. A nonempty set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in V is linearly independent if no vector in S can be written as a linear combination of the other vectors in S. Otherwise, S is linearly dependent.

Note: If $S = \{\vec{v}\}$ contains one vector, then S is linearly independent if $\vec{v} \neq \vec{0}$ and linearly dependent if $\vec{v} = \vec{0}$.

Theorem. A nonempty set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in V is linearly independent if and only if the only solution to the equation

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \vec{0}$$

is $k_1 = k_2 = \cdots = k_r = 0$.

Example 1. The set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of standard unit vectors in $\mathbb{R}^n$ is linearly independent.

Because $k_1\vec{e}_1 + k_2\vec{e}_2 + \cdots + k_n\vec{e}_n = (k_1, k_2, \cdots, k_n)$, the only solution to $k_1\vec{e}_1 + \cdots + k_n\vec{e}_n = \vec{0}$ is $k_1 = 0, k_2 = 0, \cdots, k_n = 0$.

Thus $\{\vec{e}_1, \cdots, \vec{e}_n\}$ is linearly independent in $\mathbb{R}^n$.

Example 2. Determine whether the vectors $\vec{v}_1 = (1, -2, 3)$, $\vec{v}_2 = (5, 6, -1)$, $\vec{v}_3 = (3, 2, 1)$ are linearly independent in $\mathbb{R}^3$.

$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0} \Rightarrow k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = (0,0,0)$

$$\Rightarrow \begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases} \Rightarrow k_1 = -\tfrac{1}{2}t,\ k_2 = -\tfrac{1}{2}t,\ k_3 = t.$$

or: $\begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} = \cancel{\neq} 0.$

This system has non-trivial solutions, so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

Example 3. Determine whether the vectors $\vec{v}_1 = (1, 2, 2, -1)$, $\vec{v}_2 = (4, 9, 9, -4)$, $\vec{v}_3 = (5, 8, 9, -5)$ are linearly independent in $\mathbb{R}^4$.

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0} \Rightarrow k_1(1,2,2,-1) + k_2(4,9,9,-4) + k_3(5,8,9,-5) = \vec{0}$$

$$\Rightarrow \begin{cases} k_1 + 4k_2 + 5k_3 = 0 \\ 2k_1 + 9k_2 + 8k_3 = 0 \\ 2k_1 + 9k_2 + 9k_3 = 0 \\ -k_1 - 4k_2 - 5k_3 = 0 \end{cases} \Rightarrow k_1 = 0,\ k_2 = 0,\ k_3 = 0$$

Gaussian elimination.

(note: cannot use determinant, because the coefficient matrix is not square)

Thus $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

Example 4. The set $\{1, x, x^2, \ldots, x^n\}$ of polynomials in P_n is linearly independent.

If $a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n) = 0$, then $a_0 = a_1 = \cdots = a_n = 0$.

Thus $\{1, x, x^2, \cdots, x^n\}$ is linearly independent in P_n.

Example 5. Determine whether the polynomials $p_1(x) = 1 - x$, $p_2(x) = 5 + 3x - 2x^2$, $p_3(x) = 1 + 3x - x^2$ are linearly independent in P_2.

$$k_1 p_1(x) + k_2 p_2(x) + k_3 p_3(x) = 0 \Rightarrow k_1(1-x) + k_2(5+3x-2x^2) + k_3(1+3x-x^2) = 0.$$

$$\Rightarrow \begin{cases} k_1 + 5k_2 + k_3 = 0 & \text{(from constant terms)} \\ -k_1 + 3k_2 + 3k_3 = 0 & \text{(from linear terms)} \\ -2k_2 - k_3 = 0 & \text{(from quadratic terms)} \end{cases}$$

Because $\begin{vmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{vmatrix} = 0$, this system has non-trivial solutions.

Therefore, $p_1(x), p_2(x), p_3(x)$ are linearly dependent.

Theorem. Let S be a nonempty set of vectors in a vector space V.

(a) If $\vec{0}$ is in S then S is linearly dependent.

(b) If S contains exactly two vectors, then S is linearly independent if and only if neither vector is a scalar multiple of the other.

i.e. $\vec{u} = k\vec{v} \iff \vec{u}, \vec{v}$ are linearly dependent.

Example 6. Recall that $F(-\infty, \infty)$ is the set of all functions defined on $(-\infty, \infty)$.

(a) Show that the functions $f(x) = x$ and $g(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$

$f(x)$ is not a scalar multiple of $g(x)$, so f and g are linearly independent in $F(-\infty,\infty)$.

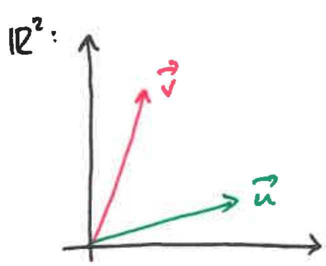
(b) Show that the functions $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$ are linearly dependent in $F(-\infty, \infty)$

$f(x) = \sin 2x = 2\sin x \cos x = 2\,g(x)$

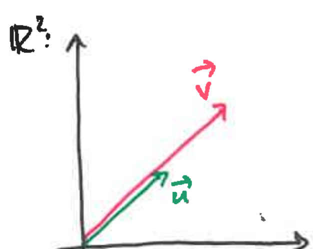
Because f is a scalar multiple of g, the functions f and g are linearly dependent in $F(-\infty,\infty)$.

The second condition in the previous theorem can be interpreted – and extended – geometrically as follows.

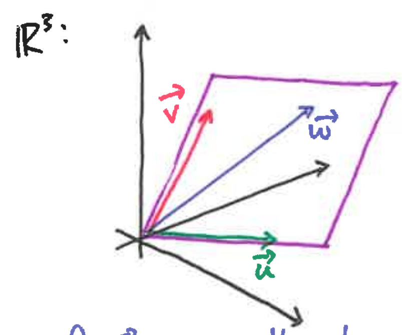
- Two distinct nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are linearly dependent if and only if they are parallel – that is, they lie on the same line.
- Three distinct nonzero vectors in $\mathbb{R}^3$ are linearly dependent if and only if they lie in the same plane.



linearly independent



linearly dependent



if $\vec{w}$ is in the plane with $\vec{u},\vec{v}$ then $\vec{u},\vec{v},\vec{w}$ are linearly dependent.

Theorem. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a nonempty set of vectors ~~vectors~~ in $\mathbb{R}^n$. If $r > n$ then S is linearly dependent.

This says that a linearly independent set in $\mathbb{R}^n$ contains at most n vectors.

eg. $\{(0,1), (2,-1), (1,3)\}$ is linearly dependent in $\mathbb{R}^2$.

note: $7(0,1) + 1(2,-1) - 2(1,3) = (0,0)$.

Our first methods of solving a linear system involved reduction of the coefficient matrix to (reduced) row echelon form. The next example demonstrates a general principle about matrices in ref and rref:

If an (augmented) matrix is in ref (rref) then the set of nonzero rows is linearly independent.

Example 7. Let $A = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 1 & a_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and let $\vec{r}_1 = (1, a_{12}, a_{13}, a_{14})$, $\vec{r}_2 = (0,0,1,a_{24})$, $\vec{r}_3 = (0,0,0,1)$.

Show that the equation $c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = \vec{0}$ has only the trivial solution $c_1 = c_2 = c_3 = 0$.

$$c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = \vec{0} \Rightarrow c_1(1, a_{12}, a_{13}, a_{14}) + c_2(0,0,1,a_{24}) + c_3(0,0,0,1) = \vec{0}$$

$$\Rightarrow \begin{cases} c_1 = 0 \\ c_1 a_{12} = 0 \\ c_1 a_{13} + c_2 = 0 \\ c_1 a_{14} + c_2 a_{24} + c_3 = 0 \end{cases} \Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0$$

Thus $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are linearly independent.

Given functions $f_1(x), f_2(x), \dots, f_n(x)$ that are differentiable $n-1$ times on $(-\infty, \infty)$, the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

"differentiate n-1 times"

is the Wronskian of $f_1, f_2, \dots, f_n$.

Theorem. If the Wronskian of the functions $f_1, f_2, \dots, f_n$ is not identically zero on $(-\infty, \infty)$, then the functions are linearly independent.

note: The converse is not true!!!
That is, if $W(x) = 0$ for all x, then $f_1, \dots, f_n$ could be either linearly independent or linearly dependent.

Example 8. Show that $f(x) = x$ and $g(x) = \cos x$ are linearly independent in $C^\infty(-\infty, \infty)$.

$$W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x\sin x - \cos x.$$

Because $W(x)$ is not identically zero (eg. $W(0) = -1$), $f(x)$ and $g(x)$ are linearly independent.

Example 9. Show that $f_1(x) = 1$, $f_2(x) = e^x$, $f_3(x) = e^{2x}$ are linearly independent in $C^\infty(-\infty, \infty)$.

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} = 4e^{3x} - 2e^{3x} = 2e^{3x}.$$

Because $W(x)$ is not identically zero (eg. $W(0) = 2$), the functions f_1, f_2, f_3 are linearly independent in $C^\infty(-\infty, \infty)$.