Section 4.3 Spanning Sets

Objectives.

- Introduce the span of a set of vectors.
- Define spanning sets for a subspace of a vector space.
- Discuss examples of spanning sets in real vector spaces.

Let V be a vector space, and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ be vectors in V. The vector $\vec{w}$ in V is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ if there are scalars $k_1, k_2, \ldots, k_r$ such that

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r.$$

Theorem. If $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_r\}$ is a nonempty set of vectors in a vector space V, then:

(a) The set W of all linear combinations of vectors in S is a subspace of V. $W = \text{Span}(S)$. (also, W is "spanned" by S).

(b) The set W in part (a) is the smallest subspace of V that contains all the vectors in S. *(This means that any other subspace of V that contains S also contains every vector in W.)*

Proof of (a). Let $\vec{u} = a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_r\vec{w}_r$, $\vec{v} = b_1\vec{w}_1 + b_2\vec{w}_2 + \cdots + b_r\vec{w}_r$.

Then: $\vec{u} + \vec{v} = (a_1+b_1)\vec{w}_1 + (a_2+b_2)\vec{w}_2 + \cdots + (a_r+b_r)\vec{w}_r$,

$k\vec{u} = (ka_1)\vec{w}_1 + (ka_2)\vec{w}_2 + \cdots + (ka_r)\vec{w}_r$.

Because W is closed under addition and scalar multiplication, W is a subspace of V.

Proof of (b). If W' is a subspace of V that contains S, then W' is closed under addition and scalar multiplication. Thus W' contains all linear combinations of vectors in S, so W' contains W.

The subspace W in this theorem is the called subspace of V spanned by S, and we say that the vectors in S span the subspace W.

Example 1. Every vector in $\mathbb{R}^n$ can be written as a <u>linear combination</u> of the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$.

$\vec{e}_1 = (1,0,0,\cdots,0)$

$\vec{e}_2 = (0,1,0,\cdots,0)$

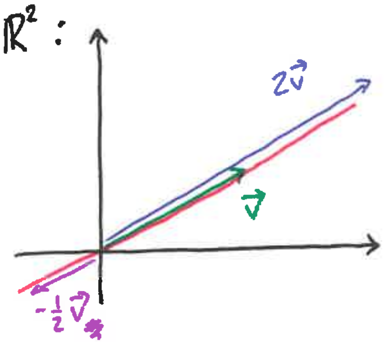
Let $\vec{v} = (v_1, v_2, \cdots, v_n)$ be a vector in $\mathbb{R}^n$.

Then $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$.

That is, $\vec{v}$ is in $\text{span}\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$.

Example 2. (a) Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. Describe $\text{span}\{\vec{v}\}$.

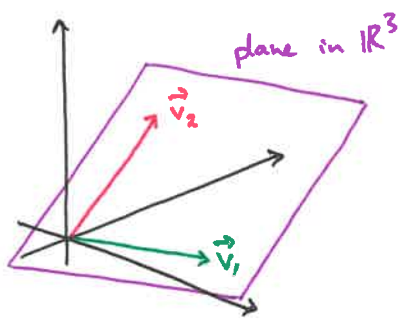
$\mathbb{R}^2$:



$\text{span}\{\vec{v}\}$ is the set of all scalar multiples of $\vec{v}$. Thus $\text{span}\{\vec{v}\}$ is the ~~line~~ line through the origin parallel to $\vec{v}$.

(b) Let $\vec{v}_1$ and $\vec{v}_2$ be non-parallel vectors in $\mathbb{R}^3$. Describe $\text{span}\{\vec{v}_1, \vec{v}_2\}$.

$\mathbb{R}^3$:



Every vector $k_1\vec{v}_1 + k_2\vec{v}_2$ lies in the plane determined by $\vec{v}_1$ and $\vec{v}_2$.

Thus $\text{span}\{\vec{v}_1, \vec{v}_2\}$ is the plane through the ~~par~~ origin and parallel to both $\vec{v}_1$ and $\vec{v}_2$.

Example 3. Every polynomial in P_n can be written as a linear combination of the polynomials $1, x, x^2, \ldots, x^n$.

all polynomials of degree $\leq n$.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ ← arbitrary polynomial in P_n

$= a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n)$.

Thus $p(x)$ is in $\text{span}\{1, x, x^2, \cdots, x^n\}$.

Therefore $P_n = \text{span}\{1, x, x^2, \ldots, x^n\}$.

There are two important problems we can ask about spanning sets in a vector space.

- Given a set of vectors S and a vector $\vec{v}$, decide whether $\vec{v}$ is in $\operatorname{span}(S)$.
- Given a set of vectors S, decide whether $\operatorname{span}(S) = V$.

← can $\vec{v}$ be written as a linear combination of vectors in S?

Example 4. Let $\vec{u} = (1, 2, -1)$ and $\vec{v} = (6, 4, 2)$.

(a) Show that $\vec{w_1} = (9, 2, 7)$ is a linear combination of $\vec{u}$ and $\vec{v}$.

The equation $(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$ is equivalent to the linear system: $9 = k_1 + 6k_2$, $2 = 2k_1 + 4k_2$, $7 = -k_1 + 2k_2$.

This system has solution $k_1 = -3$, $k_2 = 2$.

Thus $\vec{w}_1 = -3\vec{u} + 2\vec{v}$. (i.e. $\vec{w}_1$ is in $\operatorname{span}\{\vec{u}, \vec{v}\}$.)

(b) Show that $\vec{w_2} = (4, -1, 8)$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

The equation $(4,-1,8) = k_1(1,2,-1) + k_2(6,4,2)$ is equivalent to the linear system: $4 = k_1 + 6k_2$, $-1 = 2k_1 + 4k_2$, $8 = -k_1 + 2k_2$.

This system is inconsistent (i.e. no solutions!!!), so $\vec{w}_2$ is not a linear combination of $\vec{u}$ and $\vec{v}$. (i.e $\vec{w}_2$ is not in $\operatorname{span}\{\vec{u}, \vec{v}\}$.)

Example 5. Determine whether the vectors $\vec{v}_1 = (1, 1, 2)$, $\vec{v}_2 = (1, 0, 1)$, and $\vec{v}_3 = (2, 1, 3)$ span $\mathbb{R}^3$.

We need to decide whether every vector (b_1, b_2, b_3) is in $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$(b_1, b_2, b_3) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

This is equivalent to the linear system:

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned} \implies \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Because $\det \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} = 0$ (check this!!!), this system is inconsistent for some choices of b_1, b_2, b_3. Thus $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not $\mathbb{R}^3$.

Strategy. To determine whether the set $S = \{\vec{w}_1, \ldots, \vec{w}_r\}$ spans the vector space V:

- choose an arbitrary vector $\vec{v}$ in V.
- set up a linear system from $\vec{v} = k_1\vec{w}_1 + \cdots + k_r\vec{w}_r$.
- decide whether the linear system is consistent for all $\vec{v}$ in V.

Example 6. Determine whether the set S spans P_2.

(a) $S = \{1 + x + x^2, -1 - x, 2 + 2x + x^2\}$

Let $p(x) = a + bx + cx^2$. The equation

$$a + bx + cx^2 = k_1(1 + x + x^2) + k_2(-1-x) + k_3(2 + 2x + x^2)$$ is equivalent to

$$\begin{cases} k_1 - k_2 + 2k_3 = a \\ k_1 - k_2 + 2k_3 = b \\ k_1 + k_3 = c. \end{cases}$$

Because $\begin{vmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 0$, the system is inconsistent for some choices of a, b, c.

Thus S does not span P_2.

(b) $S = \{x + x^2, x - x^2, 1 + x, 1 - x\}$

Let $p(x) = a + \cancel{bx} + cx^2$. The equation

$$a + bx + cx^2 = k_1(x + x^2) + k_2(x - x^2) + k_3(1 + x) + k_4(1 - x)$$

is equivalent to the linear system

$$\begin{cases} k_3 + k_4 = a \\ k_1 + k_2 + k_3 - k_4 = b \\ k_1 - k_2 = c \end{cases} \implies \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

The rref for this system is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{-a+b+c}{2} \\ 0 & 1 & 0 & 0 & \frac{-a+b-c}{2} \\ 0 & 0 & 1 & -1 & a \end{array}\right]$.

This is consistent for all choices of a, b, c, so S spans P_2.

Example 7. Determine whether the set S spans M_{22}.

(a) $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k_1 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a = k_1 + k_2 + k_3 + k_4 \\ b = 2k_1 + 2k_3 + k_4 \\ c = k_3 + k_4 \\ d = k_1 + k_2 + k_4 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Because $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} = -2 \neq 0$, this system is consistent for <u>all</u> choices of a, b, c, d. Thus $\text{span}\{S\} = M_{22}$.

(b) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$\Rightarrow \begin{cases} a = k_1 - k_2 \\ b = k_4 \\ c = k_2 + k_3 - k_4 \\ d = k_4 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Because $\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0$ $(R_2 = R_4)$, this system is inconsistent for <u>some</u> choices of a, b, c, d.

Thus $\text{span}\{S\} \neq M_{22}$.