Section 4.2 Subspaces

Objectives.

- Introduce the notion of a subspace of a vector space.
- Determine whether a subset of a vector space is a subspace.
- Discuss some subspaces of real vector spaces.

Recall that a vector space is a set V that generalizes the vector arithmetic of $\mathbb{R}^n$ – vectors in V can be added or scaled without leaving V, and these operations are consistent with the usual rules of arithmetic.

Suppose that W is a set of vectors in a vector space V. We call W a subspace of V if W is a vector space with the operations of addition and scalar multiplication from V.

i.e. a subspace is a vector space inside a larger vector space.

Example 1. If V is any vector space, and $\vec{0}$ is the zero vector in V, then $W = \{\vec{0}\}$ is a subspace of V.

every vector in W is also in V.

why? $W \subseteq V$ and $W = \{\vec{0}\}$ is a vector space.

Six of the ten axioms for a vector space are satisfied by every subset of vectors. The four axioms that need to be checked are:

- closure under addition axiom 1
- existence of $\vec{0}$ axiom 4
- existence of negatives axiom 5
- closure under scalar multiplication. axiom 6

Subspace Test. If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if both of the following conditions are satisfied.

1. If $\vec{u}$ and $\vec{v}$ are in W, then $\vec{u} + \vec{v}$ is in W.

2. If $\vec{u}$ is in W and k is a scalar, then $k\vec{u}$ is in W

Strategy. To show that W is a subspace of V:

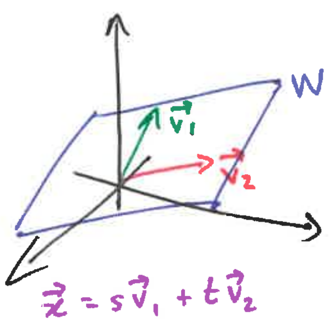
- show that if $\vec{u}_1, \vec{u}_2$ are in W, then $\vec{u}_1 + \vec{u}_2$ is in W
- show that if $\vec{u}$ is in W, then $k\vec{u}$ is in W for all k.

Example 2. If W is a line through the origin in $\mathbb{R}^n$, then W is a subspace of $\mathbb{R}^n$.

Let W be the line $\vec{x} = t\vec{v}$. If $\vec{u}_1 = s_1\vec{v}$ and $\vec{u}_2 = s_2\vec{v}$, then $\vec{u}_1 + \vec{u}_2 = s_1\vec{v} + s_2\vec{v} = (s_1 + s_2)\vec{v}$, so $\vec{u}_1 + \vec{u}_2$ is in W.

If $\vec{u} = s\vec{v}$ and k is a scalar, then $k\vec{u} = k(s\vec{v}) = (ks)\vec{v}$, so $k\vec{u}$ is in W. Because W is closed under addition and closed under scalar multiplication, W is a subspace of $\mathbb{R}^n$.

Example 3. If W is a plane through the origin in $\mathbb{R}^3$, then W is a subspace of $\mathbb{R}^3$.



If $\vec{u}_1 = s_1\vec{v}_1 + s_2\vec{v}_2$ and $\vec{u}_2 = t_1\vec{v}_1 + t_2\vec{v}_2$, then

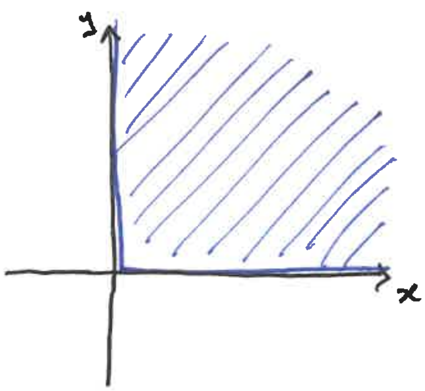
$$\vec{u}_1 + \vec{u}_2 = (s_1\vec{v}_1 + s_2\vec{v}_2) + (t_1\vec{v}_1 + t_2\vec{v}_2) = (s_1 + t_1)\vec{v}_1 + (s_2 + t_2)\vec{v}_2.$$

If $\vec{u} = s\vec{v}_1 + t\vec{v}_2$ and k is a scalar, then

$$k\vec{u} = k(s\vec{v}_1 + t\vec{v}_2) = (ks)\vec{v}_1 + (kt)\vec{v}_2.$$

Thus W is a subspace of $\mathbb{R}^3$.

Example 4. The set W of all points (x, y) in $\mathbb{R}^2$ with $x \geq 0$ and $y \geq 0$ is <u>not</u> a subspace of $\mathbb{R}^2$.



This set <u>is</u> closed under addition, but is <u>not</u> closed under scalar multiplication.

eg. $\vec{u} = (1, 1)$ is in W, but

$-1\vec{u} = (-1, -1)$ is <u>not</u> in W.

Thus W is <u>not</u> a subspace of $\mathbb{R}^2$.

Subspaces of $\mathbb{R}^2$.	**Subspaces of $\mathbb{R}^3$.**
• $\{\vec{0}\}$ • lines through the origin • $\mathbb{R}^2$	• $\{\vec{0}\}$ • lines through the origin • planes through the origin • $\mathbb{R}^3$

Recall that M_{nn} is the vector space of all $n \times n$ matrices of real numbers.

Example 5. Let W be the set of all symmetric $n \times n$ matrices.

(a) Discuss why W is a subspace of M_{nn}.

The sum of two symmetric matrices is a symmetric matrix, and a scalar multiple of a symmetric matrix is symmetric.

Thus W is a subspace of M_{nn}.

(b) What are some other subspaces of M_{nn}?

- diagonal matrices.
- upper triangular matrices.
- lower triangular matrices.

Example 6. Let W be the set of all invertible 2×2 matrices.

(a) Find two matrices A and B in W such that $A + B$ is not in W. (What does this example show?)

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are in W (because $\det \neq 0$),
but $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is <u>not</u> in W (b/c $\det(A+B) = 0$).

Thus W is <u>not</u> closed under addition, and thus ~~W~~ is <u>not</u> a subspace of M_{22}.

(b) Find a matrix A and a scalar k such that kA is not in W. (What does this example show?)

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is in W (b/c $\det A \neq 0$), but $0A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is <u>not</u> in W (b/c $\det(0A) = 0$).

Thus W is <u>not</u> closed under scalar ~~multicaption~~ multiplication, and thus is <u>not</u> a subspace of M_{22}.

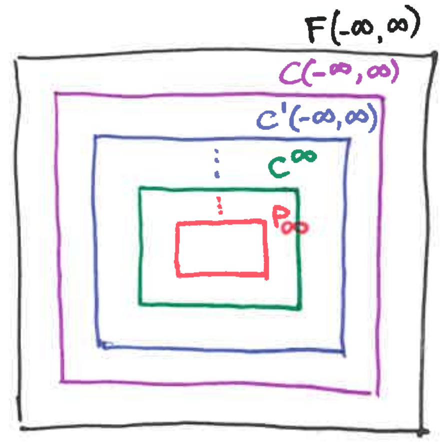
Note: more generally, the set of all invertible $n \times n$ matrices is not a subspace of M_{nn}.

Recall that $F(-\infty,\infty)$ is the set of all (real-valued) functions defined on the interval $(-\infty,\infty)$.

Example 7. The set $C(-\infty,\infty)$ of all *continuous* functions defined on $(-\infty,\infty)$ is a subspace of $F(-\infty,\infty)$.

If $f(x)$ and $g(x)$ are continuous, then $f(x)+g(x)$ and $kf(x)$ are also continuous.

Example 8. The following sets of functions are subspaces of $F(-\infty,\infty)$.



(a) $C^1(-\infty,\infty)$ • all f<u>n</u>s f where the derivative is continuous.

(b) $C^m(-\infty,\infty)$ • all f<u>n</u>s f where the first m derivatives are continuous.

m is a positive integer

(c) $C^\infty(-\infty,\infty)$ • all f<u>n</u>s f where every derivative is continuous.

A <u>polynomial of degree n</u> is a function that can be written

$$p(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$.

Example 9. The set P_∞ of all polynomials is a subspace of $F(-\infty,\infty)$.

If $p(x)$ and $q(x)$ are polynomials, then $p(x)+q(x)$ and $kp(x)$ are both polynomials. Thus P_∞ is a subspace of $F(-\infty,\infty)$.

Example 10. The set P_n of all polynomials with degree at most n is a subspace of $F(-\infty,\infty)$.

If $p(x), q(x)$ are polynomials with degree $\leq n$, then $p(x)+q(x)$ and $kp(x)$ are polynomials with degree $\leq n$.

<u>note</u>: If $p(x) = 1-2x^2$, $q(x) = 1+2x^2$, then $p(x)+q(x) = 2$ has degree < 2.

Thus the set of polynomials with degree n is <u>not</u> a subspace of $F(-\infty,\infty)$.

Example 11. Determine whether each set of matrices is a subspace of M_{22}.

(a) The set U of all matrices of the form $\begin{bmatrix} x & 2x \\ 0 & y \end{bmatrix}$. Let $A = \begin{bmatrix} a & 2a \\ 0 & b \end{bmatrix}$, $B = \begin{bmatrix} c & 2c \\ 0 & d \end{bmatrix}$. Then:

$A + B = \begin{bmatrix} a+c & 2(a+c) \\ 0 & b+d \end{bmatrix}$ is in U (take $x = a+c$, $y = b+d$), and

$kA = \begin{bmatrix} ka & 2ka \\ 0 & kb \end{bmatrix}$ is in U (take $x = ka$, $y = kb$).

Thus U is a subspace of M_{22}.

(b) The set W of all 2×2 matrices A such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Then: $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so A is in W.

But $(2A) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, so $2A$ is <u>not</u> in W.

Thus W is <u>not</u> closed under scalar multiplication, so is <u>not</u> a subspace of M_{22}.

Example 12. Determine whether each set of polynomials is a subspace of P_2.

(a) The set U of all polynomials of the form $p(x) = 1 - ax + ax^2$.

If $p(x) = 1 - x + x^2$ and $q(x) = 1 - 2x + 2x^2$, then p, q are in U

but $p(x) + q(x) = 2 - 3x + 3x^2$ is <u>not</u> in U.

Thus U is <u>not</u> a subspace of P_2.

(b) The set W of all polynomials such that $p(3) = 0$.

If p, q satisfy $p(3) = 0$ and $q(3) = 0$, then

$(p+q)(3) = p(3) + q(3) = 0 + 0 = 0$, and

$(kp)(3) = k\,p(3) = k \cdot 0 = 0$.

Thus $p+q$ and kp are in W, so W is a subspace of P_2.

Theorem. If $W_1, W_2, \ldots, W_k$ are all subspaces of a vector space V, then the set W of all vectors in the intersection of these subspaces is a subspace of V.

all vectors in every subspaces $W_1, W_2, \ldots, W_k$.

Theorem. Let A be an $m \times n$ matrix. The set of all solutions $\vec{x}$ to the homogeneous linear system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

why? If $A\vec{x}_1 = \vec{0}$ and $A\vec{x}_2 = \vec{0}$, then $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0}$
and $A(k\vec{x}_1) = k(A\vec{x}_1) = k\vec{0} = \vec{0}$.

The solution space in the previous theorem is called the kernel of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

Theorem. Let A be an $m \times n$ matrix. Then the kernel of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$.

Example 13. Describe the geometry of the solution space for each homogeneous linear system.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is $x = 2s - 3t$, $y = s$, $z = t$.
This is a plane through the origin in $\mathbb{R}^3$ with normal vector $(1, -2, 3)$.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is $x = -5t$, $y = -t$, $z = t$.
This is a line through the origin in $\mathbb{R}^3$ parallel to $\vec{v} = (-5, -1, 1)$.

(c) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is $x = 0$, $y = 0$, $z = 0$.
This is the point at the origin.

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is all (x, y, z) in $\mathbb{R}^3$.