Section 4.1 Real Vector Spaces

Objectives.

- Introduce the vector space axioms.
- Discuss some examples of real vector spaces.

A vector space is a generalization of the vector arithmetic in $\mathbb{R}^n$. A (nonempty) set of objects forms a vector space if it satisfies ten assumptions (axioms) that describe the rules of arithmetic for two operations.

real number.

Vector space axioms. Let V be a (nonempty) set of objects with two operations called *addition* and *scalar multiplication*. If the following ten axioms are satisfied by all $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and all scalars k and m, then V is a vector space.

1. If $\vec{u}$ and $\vec{v}$ are in V, then $\vec{u}+\vec{v}$ is in V. V is closed under addition.
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

   axioms 2-5 are "properties of addition"
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
4. There exists a vector $\vec{0}$ in V that satisfies $\vec{0}+\vec{u}=\vec{u}+\vec{0}=\vec{u}$ for all $\vec{u}$ in V.
5. For each $\vec{u}$ in V, the vector $-\vec{u}$ (the negative of $\vec{u}$) is in V and satisfies $\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=\vec{0}$.
6. If $\vec{u}$ is in V and k is a scalar, then $k\vec{u}$ is in V. V is closed under scalar multiplication.
7. $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
8. $(k+m)\vec{u}=k\vec{u}+m\vec{u}$

   axioms 7-10 are "properties of scalar multiplication"
9. $k(m\vec{u})=(km)\vec{u}$
10. $1\vec{u}=\vec{u}$

Strategy. To show that a set V with two operations is a vector space: usually $\mathbb{R}$

- identify the vectors and the scalars
- identify the operations of addition and scalar multiplication
- show axioms 1 and 6 hold (closure of V)
- show axioms 2-5 and axioms 7-10 hold.

Example 1. The set $V = \{\vec{0}\}$ with the operations

$$\underbrace{\vec{0}+\vec{0}=\vec{0}}_{\text{addition}} \quad \text{and} \quad \underbrace{k\vec{0}=\vec{0}}_{\text{scalar multiplication.}} \quad \text{for all scalars } k$$

is a vector space.

V is closed, because $\vec{0}+\vec{0}=\vec{0}$ is in V and $k\vec{0}=\vec{0}$ is in V.

eg. axiom 3: $\vec{0}+\underbrace{(\vec{0}+\vec{0})}_{\substack{=\vec{0} \\ \text{(by def.)}}} = \underbrace{\vec{0}}_{\substack{=\vec{0}+\vec{0} \\ \text{(by def.)}}}+\vec{0} = (\vec{0}+\vec{0})+\vec{0}$.

Example 2. The set $V = \mathbb{R}^n$ of all n-tuples of real numbers with the operations

$$\vec{u}+\vec{v} = (u_1,u_2,\ldots,u_n)+(v_1,v_2,\ldots,v_n) = (u_1+v_1,u_2+v_2,\ldots,u_n+v_n),$$
$$k\vec{u} = k(u_1,u_2,\ldots,u_n) = (ku_1,ku_2,\ldots,ku_n)$$

is a vector space.

eg. axiom 2:

$$\begin{aligned}\vec{u}+\vec{v} &= (u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n) \\ &= (u_1+v_1, u_2+v_2,\cdots,u_n+v_n) \\ &= (v_1+u_1, v_2+u_2,\cdots,v_n+u_n) \\ &= (v_1,v_2,\cdots,v_n)+(u_1,u_2,\cdots,u_n) \\ &= \vec{v}+\vec{u}.\end{aligned}$$

eg. axiom 7:

$$\begin{aligned}k(\vec{u}+\vec{v}) &= k(u_1+v_1,\cdots,u_n+v_n) \\ &= (k(u_1+v_1),\cdots,k(u_n+v_n)) \\ &= (ku_1+kv_1,\ldots,ku_n+kv_n) \\ &= (ku_1,\cdots,ku_n)+(kv_1,\cdots,kv_n) \\ &= k(u_1,\cdots,u_n)+k(v_1,\cdots,v_n) \\ &= k\vec{u}+k\vec{v}.\end{aligned}$$

Example 3. The set $V = \mathbb{R}^\infty$ of all infinite sequences of real numbers with the operations

$$\vec{u}+\vec{v} = (u_1,u_2,\ldots,u_n,\ldots)+(v_1,v_2,\ldots,v_n,\ldots) = (u_1+v_1,u_2+v_2,\ldots,u_n+v_n,\ldots),$$
$$k\vec{u} = k(u_1,u_2,\ldots,u_n,\ldots) = (ku_1,ku_2,\ldots,ku_n,\ldots)$$

is a vector space.

eg. axiom 4: Define $\vec{0}=(0,0,\cdots,0,\cdots)$. Then $\vec{0}$ is in $V=\mathbb{R}^\infty$.
If $\vec{u}=(u_1,u_2,\cdots,u_n,\cdots)$, then

$$\vec{u}+\vec{0} = (u_1,u_2,\cdots,u_n,\cdots)+(0,0,\cdots,0,\cdots) = (u_1,u_2,\cdots,u_n,\cdots) = \vec{u}$$
$$\vec{0}+\vec{u} = (0,0,\cdots,0,\cdots)+(u_1,u_2,\cdots,u_n,\cdots) = (u_1,u_2,\cdots,u_n,\cdots) = \vec{u}.$$

Example 4. The set $V = M_{22}$ of all 2×2 matrices of real numbers with the operations

$$\vec{u} + \vec{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}, \leftarrow \text{closed under addition}$$

$$k\vec{u} = k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \leftarrow \text{closed under scalar multiplication}$$

is a vector space. note: the "vectors" in M_{22} are 2x2 matrices.

axiom 5: Define $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (Then $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ for all $\vec{u}$ in M_{22}).

For $\vec{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, define $-\vec{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$. Then:

$$\vec{u} + (-\vec{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} u_{11}-u_{11} & u_{12}-u_{12} \\ u_{21}-u_{21} & u_{22}-u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}.$$

$$(-\vec{u}) + \vec{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} -u_{11}+u_{11} & -u_{12}+u_{12} \\ -u_{21}+u_{21} & -u_{22}+u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}.$$

Example 5. The set $V = M_{mn}$ of all $m \times n$ matrices of real numbers

$$\vec{u} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{bmatrix}$$

with the operations of matrix addition and scalar multiplication is a vector space.

The "vectors" in M_{mn} are $m \times n$ matrices of real numbers.

The "zero vector" in M_{34} is

$$\vec{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 6. Let $F(-\infty,\infty)$ be the set of all real-valued functions defined on the interval $(-\infty,\infty)$. For $\vec{f} = f(x)$ and $\vec{g} = g(x)$ in $F(-\infty,\infty)$, we define addition and scalar multiplication by

$$\vec{f} + \vec{g} = f(x) + g(x) \qquad \text{and} \qquad k\vec{f} = kf(x) \quad \text{for all scalars } k.$$

Then $V = F(-\infty,\infty)$ with these two operations is a vector space.

$f(x) + g(x)$ and $kf(x)$ are in $F(-\infty,\infty)$, so $F(-\infty,\infty)$ is closed under addition and scalar multiplication.

axiom 2: $\vec{f} + \vec{g} = f(x) + g(x) = g(x) + f(x) = \vec{g} + \vec{f}$.

axiom 4: Define $\vec{0} = 0$ for all x in $(-\infty,\infty)$. Then
$\vec{f} + \vec{0} = f(x) + 0 = f(x) = \vec{f}$, $\vec{0} + \vec{f} = 0 + f(x) = f(x) = \vec{f}$.

axiom 5: Define $-\vec{f} = -f(x)$. Then
$\vec{f} + (-\vec{f}) = f(x) + (-f(x)) = f(x) - f(x) = 0 = \vec{0}$.

Example 7. Let $V = \mathbb{R}^2$. For $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ in $\mathbb{R}^2$, we define addition and scalar multiplication by

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2) \qquad \text{and} \qquad k\vec{u} = (ku_1, 0).$$

Then $V = \mathbb{R}^2$ with these two operations is **not** a vector space.

This set (with the addition and multiplication defined above) satisfies axioms 1-9, but not axiom 10.

Let $\vec{u} = (u_1, u_2)$, where $u_2 \neq 0$. Then

$$1\vec{u} = \underbrace{1(u_1, u_2) = (1u_1, 0)}_{\text{by def.}} = (u_1, 0) \neq \vec{u}.$$

That is, there are vectors $\vec{u}$ in $\mathbb{R}^2$ where $1\vec{u} \neq \vec{u}$.

Therefore, this is not a vector space.

Example 8. Let V be the set of all positive real numbers. For $\vec{u} = u$ and $\vec{v} = v$ in V, we define addition and scalar multiplication by

$$\vec{u} + \vec{v} = uv \quad \text{and} \quad k\vec{u} = u^k.$$

"scalar multiplication" in V is exponentiation

Then V with these two operations is a vector space. "addition" in V is multiplication of real numbers.

If u, v positive then uv is positive. If u is positive, then u^k is positive. That is, V is closed under these two operations.

axiom 4: Define $\vec{0} = 1$. Then $\vec{u} + \vec{0} = u \cdot 1 = u = \vec{u}$.

axiom 7: For any scalar k:

$$k(\vec{u} + \vec{v}) = (uv)^k = (u^k)(v^k) = k\vec{u} + k\vec{v}$$

Some properties of vector spaces. Let V be a vector space, let $\vec{u}$ be a vector in V, and let k be a scaler. Then:

1. $0\vec{u} = \vec{0}$.
2. $k\vec{0} = \vec{0}$
3. $(-1)\vec{u} = -\vec{u}$ i.e. -1 times $\vec{u}$ equals the negative of $\vec{u}$.
4. If $k\vec{u} = \vec{0}$, then either $k = 0$ or $\vec{u} = \vec{0}$.

Proof of 1.

$$\begin{aligned}
0\vec{u} &= 0\vec{u} + \vec{0} && \text{axiom 4}\\
&= 0\vec{u} + (0\vec{u} + (-0\vec{u})) && \text{axiom 5}\\
&= (0\vec{u} + 0\vec{u}) + (-0\vec{u}) && \text{axiom 3}\\
&= (0+0)\vec{u} + (-0\vec{u}) && \text{axiom 8}\\
&= 0\vec{u} + (-0\vec{u}) && 0+0=0\\
&= \vec{0}. && \text{axiom 5}
\end{aligned}$$

Proof of 3.

We need to show that ~~[illegible]~~ $\vec{u} + (-1)\vec{u} = \vec{0}$.

i.e. $-\vec{u}$ satisfies axiom 5.

$$\begin{aligned}
\vec{u} + (-1)\vec{u} &= 1\vec{u} + (-1)\vec{u} && \text{axiom 10}\\
&= (1 + (-1))\vec{u} && \text{axiom 8}\\
&= 0\vec{u} && 1 + (-1) = 0\\
&= \vec{0} && \text{from 1.}
\end{aligned}$$