## Section 4.1 Real Vector Spaces

**Objectives.**

- Introduce the vector space axioms.
- Discuss some examples of real vector spaces.

A vector space is a generalization of the vector arithmetic in $\mathbb{R}^n$. A (nonempty) set of objects forms a vector space if it satisfies ten assumptions (axioms) that describe the rules of arithmetic for two operations.

real number.

**Vector space axioms.** Let $V$ be a (nonempty) set of objects with two operations called *addition* and *scalar multiplication*. If the following ten axioms are satisfied by all $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $V$ and all scalars $k$ and $m$, then $V$ is a vector space.

1. If $\vec{u}$ and $\vec{v}$ are in $V$, then $\vec{u}+\vec{v}$ is in $V$. V is closed under addition.
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

axioms 2-5 are "properties of addition"

3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
4. There exists a vector $\vec{0}$ in $V$ that satisfies $\vec{0}+\vec{u}=\vec{u}+\vec{0}=\vec{u}$ for all $\vec{u}$ in $V$.
5. For each $\vec{u}$ in $V$, the vector $-\vec{u}$ (the negative of $\vec{u}$) is in $V$ and satisfies $\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=\vec{0}$.
6. If $\vec{u}$ is in $V$ and $k$ is a scalar, then $k\vec{u}$ is in $V$. V is closed under scalar multiplication.
7. $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
8. $(k+m)\vec{u}=k\vec{u}+m\vec{u}$

axioms 7-10 are "properties of scalar multiplication"

9. $k(m\vec{u})=(km)\vec{u}$
10. $1\vec{u}=\vec{u}$

**Strategy.** To show that a set $V$ with two operations is a vector space: usually $\mathbb{R}$

- identify the vectors and the scalars
- identify the operations of addition and scalar multiplication
- show axioms 1 and 6 hold (closure of V)
- show axioms 2-5 and axioms 7-10 hold.

**Example 1.** The set $V = \{\vec{0}\}$ with the operations

$$\underbrace{\vec{0}+\vec{0}=\vec{0}}_{\text{addition}} \qquad \text{and} \qquad \underbrace{k\vec{0}=\vec{0} \quad \text{for all scalars } k}_{\text{scalar multiplication.}}$$

is a vector space.

$V$ is closed, because $\vec{0}+\vec{0}=\vec{0}$ is in $V$ and $k\vec{0}=\vec{0}$ is in $V$.

eg. axiom 3: $\vec{0} + \underbrace{(\vec{0}+\vec{0})}_{\substack{=\vec{0} \\ \text{(by def.)}}} = \underbrace{\vec{0}}_{\substack{=\vec{0}+\vec{0} \\ \text{(by def.)}}} + \vec{0} = (\vec{0}+\vec{0}) + \vec{0}$.

**Example 2.** The set $V = \mathbb{R}^n$ of all $n$-tuples of real numbers with the operations

$$\begin{aligned} \vec{u}+\vec{v} &= (u_1,u_2,\ldots,u_n)+(v_1,v_2,\ldots,v_n) = (u_1+v_1,u_2+v_2,\ldots,u_n+v_n), \\ k\vec{u} &= k(u_1,u_2,\ldots,u_n) = (ku_1,ku_2,\ldots,ku_n) \end{aligned}$$

is a vector space.

eg. axiom 2:

$$\begin{aligned} \vec{u}+\vec{v} &= (u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n) \\ &= (u_1+v_1, u_2+v_2, \cdots, u_n+v_n) \\ &= (v_1+u_1, v_2+u_2, \cdots, v_n+u_n) \\ &= (v_1,v_2,\cdots,v_n)+(u_1,u_2,\cdots,u_n) \\ &= \vec{v}+\vec{u}. \end{aligned}$$

eg. axiom 7:

$$\begin{aligned} k(\vec{u}+\vec{v}) &= k(u_1+v_1, \cdots, u_n+v_n) \\ &= (k(u_1+v_1), \cdots, k(u_n+v_n)) \\ &= (ku_1+kv_1, \cdots, ku_n+kv_n) \\ &= (ku_1, \cdots, ku_n)+(kv_1,\cdots,kv_n) \\ &= k(u_1,\cdots,u_n)+k(v_1,\cdots,v_n) \\ &= k\vec{u}+k\vec{v}. \end{aligned}$$

**Example 3.** The set $V = \mathbb{R}^\infty$ of all infinite sequences of real numbers with the operations

$$\begin{aligned} \vec{u}+\vec{v} &= (u_1,u_2,\ldots,u_n,\ldots)+(v_1,v_2,\ldots,v_n,\ldots) = (u_1+v_1,u_2+v_2,\ldots,u_n+v_n,\ldots), \\ k\vec{u} &= k(u_1,u_2,\ldots,u_n,\ldots) = (ku_1,ku_2,\ldots,ku_n,\ldots) \end{aligned}$$

is a vector space.

eg. axiom 4: Define $\vec{0} = (0,0,\cdots,0,\cdots)$. Then $\vec{0}$ is in $V=\mathbb{R}^\infty$.

If $\vec{u} = (u_1,u_2,\cdots,u_n,\cdots)$, then

$$\vec{u}+\vec{0} = (u_1,u_2,\cdots,u_n,\cdots)+(0,0,\cdots,0,\cdots) = (u_1,u_2,\cdots,u_n,\cdots) = \vec{u}$$

$$\vec{0}+\vec{u} = (0,0,\cdots,0,\cdots)+(u_1,u_2,\cdots,u_n,\cdots) = (u_1,u_2,\cdots,u_n,\cdots) = \vec{u}.$$

**Example 4.** The set $V = M_{22}$ of all $2 \times 2$ matrices of real numbers with the operations

$$\vec{u} + \vec{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}, \leftarrow \text{closed under addition}$$

$$k\vec{u} = k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \quad \leftarrow \text{closed under scalar multiplication}$$

is a vector space. note: the "vectors" in $M_{22}$ are 2x2 matrices.

axiom 5: Define $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (Then $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ for all $\vec{u}$ in $M_{22}$).

For $\vec{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, define $-\vec{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$. Then:

$$\vec{u} + (-\vec{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} - u_{11} & u_{12} - u_{12} \\ u_{21} - u_{21} & u_{22} - u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}.$$

$$(-\vec{u}) + \vec{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} -u_{11} + u_{11} & -u_{12} + u_{12} \\ -u_{21} + u_{21} & -u_{22} + u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}.$$

**Example 5.** The set $V = M_{mn}$ of all $m \times n$ matrices of real numbers

$$\vec{u} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{bmatrix}$$

with the operations of matrix addition and scalar multiplication is a vector space.

The "vectors" in $M_{mn}$ are $m \times n$ matrices of real numbers.

The "zero vector" in $M_{34}$ is

$$\vec{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

note: $F(a,b)$ is all functions defined on $(a,b)$.
$F[a,b]$ is all functions defined on $[a,b]$.



**Example 6.** Let $F(-\infty,\infty)$ be the set of all real-valued functions defined on the interval $(-\infty,\infty)$. For $\vec{f} = f(x)$ and $\vec{g} = g(x)$ in $F(-\infty,\infty)$, we define addition and scalar multiplication by

$$\vec{f} + \vec{g} = f(x) + g(x) \qquad \text{and} \qquad k\vec{f} = kf(x) \quad \text{for all scalars } k.$$

Then $V = F(-\infty,\infty)$ with these two operations is a vector space.

$f(x) + g(x)$ and $kf(x)$ are in $F(-\infty,\infty)$, so $F(-\infty,\infty)$ is closed under addition and scalar multiplication.

axiom 2: $\vec{f} + \vec{g} = f(x) + g(x) = g(x) + f(x) = \vec{g} + \vec{f}$.

axiom 4: Define $\vec{0} = 0$ for all $x$ in $(-\infty,\infty)$. Then
$\vec{f} + \vec{0} = f(x) + 0 = f(x) = \vec{f}$, $\vec{0} + \vec{f} = 0 + f(x) = f(x) = \vec{f}$.

axiom 5: Define $-\vec{f} = -f(x)$. Then
$\vec{f} + (-\vec{f}) = f(x) + (-f(x)) = f(x) - f(x) = 0 = \vec{0}$.

**Example 7.** Let $V = \mathbb{R}^2$. For $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ in $\mathbb{R}^2$, we define addition and scalar multiplication by

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2) \qquad \text{and} \qquad k\vec{u} = (ku_1, 0).$$

Then $V = \mathbb{R}^2$ with these two operations is **not** a vector space.

This set (with the addition and multiplication defined above) satisfies axioms 1-9, but not axiom 10.

Let $\vec{u} = (u_1, u_2)$, where $u_2 \neq 0$. Then

$$1\vec{u} = \underbrace{1(u_1,u_2) = (1u_1, 0)}_{\text{by def.}} = (u_1, 0) \neq \vec{u}.$$

That is, there are vectors $\vec{u}$ in $\mathbb{R}^2$ where $1\vec{u} \neq \vec{u}$.

Therefore, this is not a vector space.

**Example 8.** Let $V$ be the set of all positive real numbers. For $\vec{u} = u$ and $\vec{v} = v$ in $V$, we define addition and scalar multiplication by

$$\vec{u} + \vec{v} = uv \quad \text{and} \quad k\vec{u} = u^k.$$

"scalar multiplication" in V is exponentiation

Then $V$ with these two operations is a vector space. "addition" in V is multiplication of real numbers.

If u, v positive then uv is positive. If u is positive, then $u^k$ is positive.
That is, V is closed under these two operations.

axiom 4: Define $\vec{0} = 1$. Then $\vec{u} + \vec{0} = u \cdot 1 = u = \vec{u}$.

axiom 7: For any scalar k:

$$k(\vec{u} + \vec{v}) = (uv)^k = (u^k)(v^k) = k\vec{u} + k\vec{v}$$

**Some properties of vector spaces.** Let $V$ be a vector space, let $\vec{u}$ be a vector in $V$, and let $k$ be a scaler. Then:

1. $0\vec{u} = \vec{0}$.
2. $k\vec{0} = \vec{0}$
3. $(-1)\vec{u} = -\vec{u}$ i.e. $-1$ times $\vec{u}$ equals the negative of $\vec{u}$.
4. If $k\vec{u} = \vec{0}$, then either $k = 0$ or $\vec{u} = \vec{0}$.

**Proof of 1.**

$$
\begin{aligned}
0\vec{u} &= 0\vec{u} + \vec{0} && \text{axiom 4} \\
&= 0\vec{u} + (0\vec{u} + (-0\vec{u})) && \text{axiom 5} \\
&= (0\vec{u} + 0\vec{u}) + (-0\vec{u}) && \text{axiom 3} \\
&= (0+0)\vec{u} + (-0\vec{u}) && \text{axiom 8} \\
&= 0\vec{u} + (-0\vec{u}) && 0+0=0 \\
&= \vec{0}. && \text{axiom 5}
\end{aligned}
$$

**Proof of 3.**

We need to show that $\vec{u} + (-1)\vec{u} = \vec{0}$.

i.e. $-\vec{u}$ satisfies axiom 5.

$$
\begin{aligned}
\vec{u} + (-1)\vec{u} &= 1\vec{u} + (-1)\vec{u} && \text{axiom 10} \\
&= (1 + (-1))\vec{u} && \text{axiom 8} \\
&= 0\vec{u} && 1 + (-1) = 0 \\
&= \vec{0} && \text{from 1.}
\end{aligned}
$$

## Section 4.2 Subspaces

**Objectives.**

- Introduce the notion of a subspace of a vector space.
- Determine whether a subset of a vector space is a subspace.
- Discuss some subspaces of real vector spaces.

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Recall that a vector space is a set $V$ that generalizes the vector arithmetic of $\mathbb{R}^n$ – vectors in $V$ can be added or scaled without leaving $V$, and these operations are consistent with the usual rules of arithmetic.

Suppose that $W$ is a set of vectors in a vector space $V$. We call $W$ a subspace of $V$ if $W$ is a vector space with the operations of addition and scalar multiplication from $V$.

i.e. a subspace is a vector space inside a larger vector space.

**Example 1.** If $V$ is any vector space, and $\vec{0}$ is the zero vector in $V$, then $W = \left\{\vec{0}\right\}$ is a subspace of $V$.

every vector in W is also in V.

why? $W \subseteq V$ and $W = \{\vec{0}\}$ is a vector space.

Six of the ten axioms for a vector space are satisfied by every subset of vectors. The four axioms that need to be checked are:

- closure under addition axiom 1
- existence of $\vec{0}$ axiom 4
- existence of negatives axiom 5
- closure under scalar multiplication. axiom 6

**Subspace Test.** If $W$ is a nonempty set of vectors in a vector space $V$, then $W$ is a subspace of $V$ if and only if both of the following conditions are satisfied.

1. If $\vec{u}$ and $\vec{v}$ are in $W$, then $\vec{u} + \vec{v}$ is in $W$.
2. If $\vec{u}$ is in $W$ and $k$ is a scalar, then $k\vec{u}$ is in $W$

**Strategy.** To show that $W$ is a subspace of $V$:

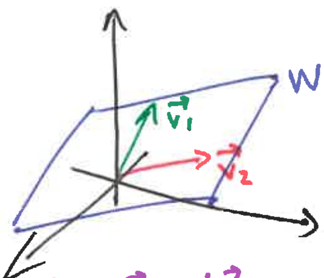
- show that if $\vec{u}_1, \vec{u}_2$ are in $W$, then $\vec{u}_1 + \vec{u}_2$ is in $W$
- show that if $\vec{u}$ is in $W$, then $k\vec{u}$ is in $W$ for all $k$.

**Example 2.** If $W$ is a line through the origin in $\mathbb{R}^n$, then $W$ is a subspace of $\mathbb{R}^n$.

Let $W$ be the line $\vec{x} = t\vec{v}$. If $\vec{u}_1 = s_1\vec{v}$ and $\vec{u}_2 = s_2\vec{v}$, then $\vec{u}_1 + \vec{u}_2 = s_1\vec{v} + s_2\vec{v} = (s_1 + s_2)\vec{v}$, so $\vec{u}_1 + \vec{u}_2$ is in $W$.

If $\vec{u} = s\vec{v}$ and $k$ is a scalar, then $k\vec{u} = k(s\vec{v}) = (ks)\vec{v}$, so $k\vec{u}$ is in $W$. Because $W$ is closed under addition and closed under scalar multiplication, $W$ is a subspace of $\mathbb{R}^n$.

**Example 3.** If $W$ is a plane through the origin in $\mathbb{R}^3$, then $W$ is a subspace of $\mathbb{R}^3$.




$\vec{x} = s\vec{v}_1 + t\vec{v}_2$

If $\vec{u}_1 = s_1\vec{v}_1 + s_2\vec{v}_2$ and $\vec{u}_2 = t_1\vec{v}_1 + t_2\vec{v}_2$, then

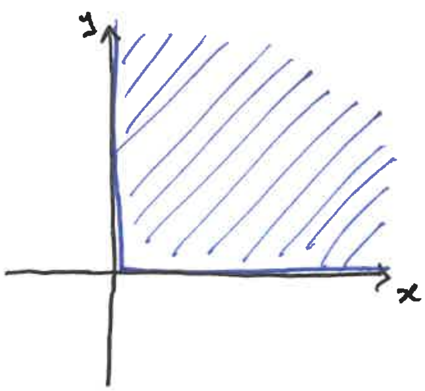
$\vec{u}_1 + \vec{u}_2 = (s_1\vec{v}_1 + s_2\vec{v}_2) + (t_1\vec{v}_1 + t_2\vec{v}_2) = (s_1 + t_1)\vec{v}_1 + (s_2 + t_2)\vec{v}_2$.

If $\vec{u} = s\vec{v}_1 + t\vec{v}_2$ and $k$ is a scalar, then

$k\vec{u} = k(s\vec{v}_1 + t\vec{v}_2) = (ks)\vec{v}_1 + (kt)\vec{v}_2$.

Thus $W$ is a subspace of $\mathbb{R}^3$.

**Example 4.** The set $W$ of all points $(x, y)$ in $\mathbb{R}^2$ with $x \geq 0$ and $y \geq 0$ is <u>not</u> a subspace of $\mathbb{R}^2$.




This set <u>is</u> closed under addition, but is <u>not</u> closed under scalar multiplication.

eg. $\vec{u} = (1,1)$ is in $W$, but

$-1\vec{u} = (-1,-1)$ is <u>not</u> in $W$.

Thus $W$ is <u>not</u> a subspace of $\mathbb{R}^2$.

**Subspaces of $\mathbb{R}^2$.**

- $\{\vec{0}\}$
- lines through the origin
- $\mathbb{R}^2$

**Subspaces of $\mathbb{R}^3$.**

- $\{\vec{0}\}$
- lines through the origin
- planes through the origin
- $\mathbb{R}^3$

Recall that $M_{nn}$ is the vector space of all $n \times n$ matrices of real numbers.

**Example 5.** Let $W$ be the set of all symmetric $n \times n$ matrices.

(a) Discuss why $W$ is a subspace of $M_{nn}$.

The sum of two symmetric matrices is a symmetric matrix, and a scalar multiple of a symmetric matrix is symmetric.

Thus $W$ is a subspace of $M_{nn}$.

(b) What are some other subspaces of $M_{nn}$?

- diagonal matrices.
- upper triangular matrices.
- lower triangular matrices.

**Example 6.** Let $W$ be the set of all invertible $2 \times 2$ matrices.

(a) Find two matrices $A$ and $B$ in $W$ such that $A + B$ is not in $W$. (What does this example show?)

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are in $W$ (because $\det \neq 0$),

but $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is <u>not</u> in $W$ (b/c $\det(A+B) = 0$).

Thus $W$ is <u>not</u> closed under addition, and thus ~~W~~ is <u>not</u> a subspace of $M_{22}$.

(b) Find a matrix $A$ and a scalar $k$ such that $kA$ is not in $W$. (What does this example show?)

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is in $W$ (b/c $\det A \neq 0$), but $0A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is <u>not</u> in $W$ (b/c $\det(0A) = 0$).

Thus $W$ is <u>not</u> closed under scalar ~~multicaption~~ multiplication, and thus is <u>not</u> a subspace of $M_{22}$.

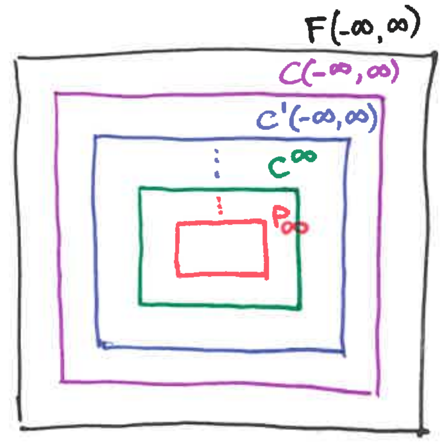
*Note: more generally, the set of all invertible $n \times n$ matrices is not a subspace of $M_{nn}$.*

Recall that $F(-\infty,\infty)$ is the set of all (real-valued) functions defined on the interval $(-\infty,\infty)$.

**Example 7.** The set $C(-\infty,\infty)$ of all *continuous* functions defined on $(-\infty,\infty)$ is a subspace of $F(-\infty,\infty)$.

If $f(x)$ and $g(x)$ are continuous, then $f(x)+g(x)$ and $kf(x)$ are also continuous.

**Example 8.** The following sets of functions are subspaces of $F(-\infty,\infty)$.




(a) $C^1(-\infty,\infty)$ • all fns $f$ where the derivative is continuous.

(b) $C^m(-\infty,\infty)$ • all fns $f$ where the first $m$ derivatives are continuous.

$m$ is a positive integer

(c) $C^\infty(-\infty,\infty)$ • all fns $f$ where every derivative is continuous.

A <u>polynomial of degree $n$</u> is a function that can be written

$$p(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$.

**Example 9.** The set $P_\infty$ of all polynomials is a subspace of $F(-\infty,\infty)$.

If $p(x)$ and $q(x)$ are polynomials, then $p(x)+q(x)$ and $kp(x)$ are both polynomials. Thus $P_\infty$ is a subspace of $F(-\infty,\infty)$.

**Example 10.** The set $P_n$ of all polynomials with degree at most $n$ is a subspace of $F(-\infty,\infty)$.

If $p(x), q(x)$ are polynomials with degree $\leq n$, then $p(x)+q(x)$ and $kp(x)$ are polynomials with degree $\leq n$.

<u>note</u>: If $p(x) = 1-2x^2$, $q(x) = 1+2x^2$, then $p(x)+q(x) = 2$ has degree $< 2$.

Thus the set of polynomials with degree $n$ is <u>not</u> a subspace of $F(-\infty,\infty)$.

**Example 11.** Determine whether each set of matrices is a subspace of $M_{22}$.

(a) The set $U$ of all matrices of the form $\begin{bmatrix} x & 2x \\ 0 & y \end{bmatrix}$. Let $A = \begin{bmatrix} a & 2a \\ 0 & b \end{bmatrix}$, $B = \begin{bmatrix} c & 2c \\ 0 & d \end{bmatrix}$. Then:

$A + B = \begin{bmatrix} a+c & 2(a+c) \\ 0 & b+d \end{bmatrix}$ is in $U$ (take $x = a+c$, $y = b+d$), and

$kA = \begin{bmatrix} ka & 2ka \\ 0 & kb \end{bmatrix}$ is in $U$ (take $x = ka$, $y = kb$).

Thus $U$ is a subspace of $M_{22}$.

(b) The set $W$ of all $2 \times 2$ matrices $A$ such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Then: $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $A$ is in $W$.

But $(2A) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, so $2A$ is <u>not</u> in $W$.

Thus $W$ is <u>not</u> closed under scalar multiplication, so is <u>not</u> a subspace of $M_{22}$.

**Example 12.** Determine whether each set of polynomials is a subspace of $P_2$.

(a) The set $U$ of all polynomials of the form $p(x) = 1 - ax + ax^2$.

If $p(x) = 1 - x + x^2$ and $q(x) = 1 - 2x + 2x^2$, then $p, q$ are in $U$

but $p(x) + q(x) = 2 - 3x + 3x^2$ is <u>not</u> in $U$.

Thus $U$ is <u>not</u> a subspace of $P_2$.

(b) The set $W$ of all polynomials such that $p(3) = 0$.

If $p, q$ satisfy $p(3) = 0$ and $q(3) = 0$, then

$(p+q)(3) = p(3) + q(3) = 0 + 0 = 0$, and

$(kp)(3) = k\,p(3) = k \cdot 0 = 0$.

Thus $p+q$ and $kp$ are in $W$, so $W$ is a subspace of $P_2$.

**Theorem.** If $W_1, W_2, \ldots, W_k$ are all subspaces of a vector space $V$, then the set $W$ of all vectors in the intersection of these subspaces is a subspace of $V$.

all vectors in every subspaces $W_1, W_2, \ldots, W_k$.

**Theorem.** Let $A$ be an $m \times n$ matrix. The set of all solutions $\vec{x}$ to the homogeneous linear system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

why? If $A\vec{x}_1 = \vec{0}$ and $A\vec{x}_2 = \vec{0}$, then $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0}$ and $A(k\vec{x}_1) = k(A\vec{x}_1) = k\vec{0} = \vec{0}$.

The solution space in the previous theorem is called the kernel of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

**Theorem.** Let $A$ be an $m \times n$ matrix. Then the kernel of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$.

**Example 13.** Describe the geometry of the solution space for each homogeneous linear system.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is $x = 2s - 3t$, $y = s$, $z = t$. This is a plane through the origin in $\mathbb{R}^3$ with normal vector $(1, -2, 3)$.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is $x = -5t$, $y = -t$, $z = t$. This is a line through the origin in $\mathbb{R}^3$ parallel to $\vec{v} = (-5, -1, 1)$.

(c) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is $x = 0$, $y = 0$, $z = 0$. This is the point at the origin.

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is all $(x, y, z)$ in $\mathbb{R}^3$.

**Section 4.3 Spanning Sets**

**Objectives.**

- Introduce the span of a set of vectors.
- Define spanning sets for a subspace of a vector space.
- Discuss examples of spanning sets in real vector spaces.

---

Let $V$ be a vector space, and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ be vectors in $V$. The vector $\vec{w}$ in $V$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ if there are scalars $k_1, k_2, \ldots, k_r$ such that

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r.$$

**Theorem.** If $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_r\}$ is a nonempty set of vectors in a vector space $V$, then:

(a) The set $W$ of all linear combinations of vectors in $S$ is a subspace of $V$. $W = \text{span}(S)$.
(also, $W$ is "spanned" by $S$).

(b) The set $W$ in part (a) is the smallest subspace of $V$ that contains all the vectors in $S$.
*(This means that any other subspace of $V$ that contains $S$ also contains every vector in $W$.)*

**Proof of (a).** Let $\vec{u} = a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_r\vec{w}_r$, $\vec{v} = b_1\vec{w}_1 + b_2\vec{w}_2 + \cdots + b_r\vec{w}_r$.

Then: $\vec{u} + \vec{v} = (a_1 + b_1)\vec{w}_1 + (a_2 + b_2)\vec{w}_2 + \cdots + (a_r + b_r)\vec{w}_r$,

$k\vec{u} = (ka_1)\vec{w}_1 + (ka_2)\vec{w}_2 + \cdots + (ka_r)\vec{w}_r$.

Because $W$ is closed under addition and scalar multiplication, $W$ is a subspace of $V$.

**Proof of (b).** If $W'$ is a subspace of $V$ that contains $S$, then $W'$ is closed under addition and scalar multiplication. Thus $W'$ contains all linear combinations of vectors in $S$, so $W'$ contains $W$.

The subspace $W$ in this theorem is the called subspace of $V$ spanned by $S$, and we say that the vectors in $S$ span the subspace $W$.

**Example 1.** Every vector in $\mathbb{R}^n$ can be written as a <u>linear combination</u> of the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$.

$\vec{e}_1 = (1, 0, 0, \cdots, 0)$

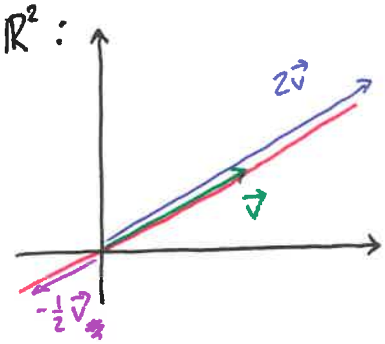
$\vec{e}_2 = (0, 1, 0, \cdots, 0)$

Let $\vec{v} = (v_1, v_2, \cdots, v_n)$ be a vector in $\mathbb{R}^n$.

Then $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$.

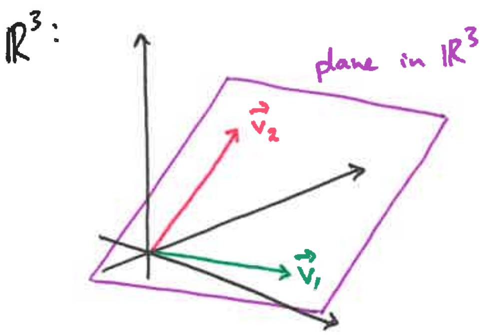
That is, $\vec{v}$ is in $\operatorname{span}\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$.

**Example 2.** (a) Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. Describe $\operatorname{span}\{\vec{v}\}$.




$\operatorname{span}\{\vec{v}\}$ is the set of all scalar multiples of $\vec{v}$. Thus $\operatorname{span}\{\vec{v}\}$ is the line through the origin parallel to $\vec{v}$.

(b) Let $\vec{v}_1$ and $\vec{v}_2$ be non-parallel vectors in $\mathbb{R}^3$. Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.




Every vector $k_1\vec{v}_1 + k_2\vec{v}_2$ lies in the plane determined by $\vec{v}_1$ and $\vec{v}_2$.

Thus $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$ is the plane through the origin and parallel to both $\vec{v}_1$ and $\vec{v}_2$.

**Example 3.** Every polynomial in $P_n$ can be written as a linear combination of the polynomials $1, x, x^2, \ldots, x^n$.

all polynomials of degree $\leq n$.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ ← arbitrary polynomial in $P_n$

$= a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n)$.

Thus $p(x)$ is in $\operatorname{span}\{1, x, x^2, \cdots, x^n\}$.

Therefore $P_n = \operatorname{span}\{1, x, x^2, \ldots, x^n\}$.

There are two important problems we can ask about spanning sets in a vector space.

- Given a set of vectors $S$ and a vector $\vec{v}$, decide whether $\vec{v}$ is in $\operatorname{span}(S)$.
- Given a set of vectors $S$, decide whether $\operatorname{span}(S) = V$.

← can $\vec{v}$ be written as a linear combination of vectors in $S$?

**Example 4.** Let $\vec{u} = (1, 2, -1)$ and $\vec{v} = (6, 4, 2)$.

(a) Show that $\vec{w_1} = (9, 2, 7)$ is a linear combination of $\vec{u}$ and $\vec{v}$.

The equation $(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$ is equivalent to the linear system: $9 = k_1 + 6k_2$, $2 = 2k_1 + 4k_2$, $7 = -k_1 + 2k_2$.

This system has solution $k_1 = -3$, $k_2 = 2$.

Thus $\vec{w}_1 = -3\vec{u} + 2\vec{v}$. (i.e. $\vec{w}_1$ is in $\operatorname{span}\{\vec{u}, \vec{v}\}$.)

(b) Show that $\vec{w_2} = (4, -1, 8)$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

The equation $(4,-1,8) = k_1(1,2,-1) + k_2(6,4,2)$ is equivalent to the linear system: $4 = k_1 + 6k_2$, $-1 = 2k_1 + 4k_2$, $8 = -k_1 + 2k_2$.

This system is inconsistent (i.e. no solutions!!!), so $\vec{w}_2$ is not a linear combination of $\vec{u}$ and $\vec{v}$. (i.e $\vec{w}_2$ is not in $\operatorname{span}\{\vec{u}, \vec{v}\}$.)

**Example 5.** Determine whether the vectors $\vec{v}_1 = (1, 1, 2)$, $\vec{v}_2 = (1, 0, 1)$, and $\vec{v}_3 = (2, 1, 3)$ span $\mathbb{R}^3$.

We need to decide whether every vector $(b_1, b_2, b_3)$ is in $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$(b_1, b_2, b_3) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3) = (k_1 + k_2 + 2k_3,\ k_1 + k_3,\ 2k_1 + k_2 + 3k_3)$$

This is equivalent to the linear system:

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \quad\ + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Because $\det\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} = 0$ (check this!!!), this system is inconsistent for some choices of $b_1, b_2, b_3$. Thus $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not $\mathbb{R}^3$.

**Strategy.** To determine whether the set $S = \{\vec{w}_1, \ldots, \vec{w}_r\}$ spans the vector space $V$:

- choose an arbitrary vector $\vec{v}$ in $V$.
- set up a linear system from $\vec{v} = k_1\vec{w}_1 + \cdots + k_r\vec{w}_r$.
- decide whether the linear system is consistent for all $\vec{v}$ in $V$.

**Example 6.** Determine whether the set $S$ spans $P_2$.

(a) $S = \{1 + x + x^2, -1 - x, 2 + 2x + x^2\}$

Let $p(x) = a + bx + cx^2$. The equation

$a + bx + cx^2 = k_1(1 + x + x^2) + k_2(-1-x) + k_3(2 + 2x + x^2)$ is equivalent to

$$\begin{cases} k_1 - k_2 + 2k_3 = a \\ k_1 - k_2 + 2k_3 = b \\ k_1 + k_3 = c. \end{cases}$$

Because $\begin{vmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 0$, the system is inconsistent for some choices of $a, b, c$.

Thus $S$ does not span $P_2$.

(b) $S = \{x + x^2, x - x^2, 1 + x, 1 - x\}$

Let $p(x) = a + bx + cx^2$. The equation

$a + bx + cx^2 = k_1(x + x^2) + k_2(x - x^2) + k_3(1 + x) + k_4(1 - x)$

is equivalent to the linear system

$$\begin{cases} k_3 + k_4 = a \\ k_1 + k_2 + k_3 - k_4 = b \\ k_1 - k_2 = c \end{cases} \Rightarrow \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

The rref for this system is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{-a+b+c}{2} \\ 0 & 1 & 0 & 0 & \frac{-a+b-c}{2} \\ 0 & 0 & 1 & -1 & a \end{array}\right]$.

This is consistent for all choices of $a, b, c$, so $S$ spans $P_2$.

**Example 7.** Determine whether the set $S$ spans $M_{22}$.

(a) $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k_1 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a = k_1 + k_2 + k_3 + k_4 \\ b = 2k_1 + 2k_3 + k_4 \\ c = k_3 + k_4 \\ d = k_1 + k_2 + k_4 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Because $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} = -2 \neq 0$, this system is consistent for all choices of $a, b, c, d$. Thus $\operatorname{span}\{S\} = M_{22}$.

(b) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$\Rightarrow \begin{cases} a = k_1 - k_2 \\ b = k_4 \\ c = k_2 + k_3 - k_4 \\ d = k_4 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

~~Th~~ Because $\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0$ $(R_2 = R_4)$, this system is inconsistent for some choices of $a, b, c, d$.

Thus $\operatorname{span}\{S\} \neq M_{22}$.

## Section 4.4 Linear Independence

**Objectives.**

- Define linear independence of vectors.
- Determine whether a set of vectors is linearly independent or linearly dependent.
- Define and apply the Wronskian to determine whether a set of functions is linearly independent.

---

Let $V$ be a vector space. A nonempty set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in $V$ is linearly independent if no vector in $S$ can be written as a linear combination of the other vectors in $S$. Otherwise, $S$ is linearly dependent.

*Note: If $S = \{\vec{v}\}$ contains one vector, then $S$ is linearly independent if $\vec{v} \neq \vec{0}$ and linearly dependent if $\vec{v} = \vec{0}$.*

**Theorem.** A nonempty set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in $V$ is linearly independent if and only if the only solution to the equation

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \vec{0}$$

is $k_1 = k_2 = \cdots = k_r = 0$.

**Example 1.** The set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of standard unit vectors in $\mathbb{R}^n$ is linearly independent.

Because $k_1\vec{e}_1 + k_2\vec{e}_2 + \cdots + k_n\vec{e}_n = (k_1, k_2, \cdots, k_n)$, the only solution to $k_1\vec{e}_1 + \cdots + k_n\vec{e}_n = \vec{0}$ is $k_1 = 0, k_2 = 0, \cdots, k_n = 0$.

Thus $\{\vec{e}_1, \cdots, \vec{e}_n\}$ is linearly independent in $\mathbb{R}^n$.

**Example 2.** Determine whether the vectors $\vec{v}_1 = (1, -2, 3)$, $\vec{v}_2 = (5, 6, -1)$, $\vec{v}_3 = (3, 2, 1)$ are linearly independent in $\mathbb{R}^3$.

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0} \Rightarrow k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = (0,0,0)$$

$$\Rightarrow \begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases} \Rightarrow k_1 = -\tfrac{1}{2}t,\ k_2 = -\tfrac{1}{2}t,\ k_3 = t.$$

or : $\begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} \overset{=}{\cancel{\neq}} 0.$

This system has non-trivial solutions, so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

**Example 3.** Determine whether the vectors $\vec{v}_1 = (1, 2, 2, -1)$, $\vec{v}_2 = (4, 9, 9, -4)$, $\vec{v}_3 = (5, 8, 9, -5)$ are linearly independent in $\mathbb{R}^4$.

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0} \Rightarrow k_1(1,2,2,-1) + k_2(4,9,9,-4) + k_3(5,8,9,-5) = \vec{0}$$

$$\Rightarrow \begin{cases} k_1 + 4k_2 + 5k_3 = 0 \\ 2k_1 + 9k_2 + 8k_3 = 0 \\ 2k_1 + 9k_2 + 9k_3 = 0 \\ -k_1 - 4k_2 - 5k_3 = 0 \end{cases} \Rightarrow k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

Gaussian elimination.

(note: cannot use determinant, because the coefficient matrix is not square)

Thus $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

**Example 4.** The set $\{1, x, x^2, \ldots, x^n\}$ of polynomials in $P_n$ is linearly independent.

If $a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n) = 0$, then $a_0 = a_1 = \cdots = a_n = 0$.

Thus $\{1, x, x^2, \cdots, x^n\}$ is linearly independent in $P_n$.

**Example 5.** Determine whether the polynomials $p_1(x) = 1 - x$, $p_2(x) = 5 + 3x - 2x^2$, $p_3(x) = 1 + 3x - x^2$ are linearly independent in $P_2$.

$$k_1 p_1(x) + k_2 p_2(x) + k_3 p_3(x) = 0 \Rightarrow k_1(1-x) + k_2(5+3x-2x^2) + k_3(1+3x-x^2) = 0.$$

$$\Rightarrow \begin{cases} k_1 + 5k_2 + k_3 = 0 & \text{(from constant terms)} \\ -k_1 + 3k_2 + 3k_3 = 0 & \text{(from linear terms)} \\ -2k_2 - k_3 = 0 & \text{(from quadratic terms)} \end{cases}$$

Because $\begin{vmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{vmatrix} = 0$, this system has non-trivial solutions.

Therefore, $p_1(x), p_2(x), p_3(x)$ are linearly dependent.

**Theorem.** Let $S$ be a nonempty set of vectors in a vector space $V$.

(a) If $\vec{0}$ is in $S$ then $S$ is linearly dependent.

(b) If $S$ contains exactly two vectors, then $S$ is linearly independent if and only if neither vector is a scalar multiple of the other.

i.e. $\vec{u} = k\vec{v} \iff \vec{u}, \vec{v}$ are linearly dependent.

**Example 6.** Recall that $F(-\infty, \infty)$ is the set of all functions defined on $(-\infty, \infty)$.

(a) Show that the functions $f(x) = x$ and $g(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$

$f(x)$ is not a scalar multiple of $g(x)$, so $f$ and $g$ are linearly independent in $F(-\infty,\infty)$.

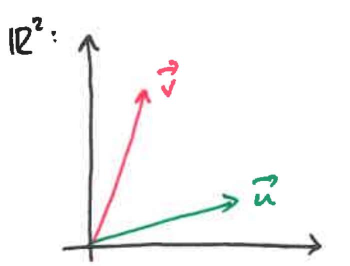
(b) Show that the functions $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$ are linearly dependent in $F(-\infty, \infty)$

$f(x) = \sin 2x = 2 \sin x \cos x = 2\, g(x)$

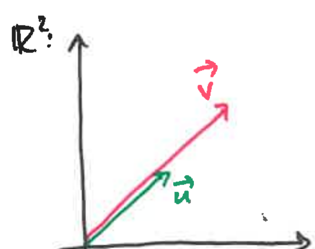
Because $f$ is a scalar multiple of $g$, the functions $f$ and $g$ are linearly dependent in $F(-\infty,\infty)$.

The second condition in the previous theorem can be interpreted – and extended – geometrically as follows.

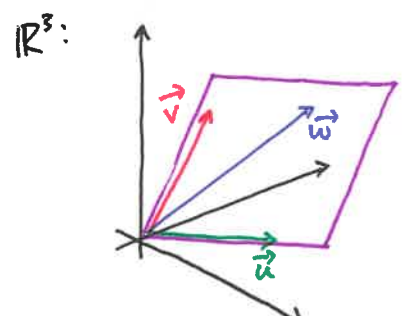
- Two distinct nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are linearly dependent if and only if they are parallel – that is, they lie on the same line.
- Three distinct nonzero vectors in $\mathbb{R}^3$ are linearly dependent if and only if they lie in the same plane.




linearly independent




linearly dependent




if $\vec{w}$ is in the plane with $\vec{u}, \vec{v}$ then $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent.

**Theorem.** Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a nonempty set of vectors ~~vectors~~ in $\mathbb{R}^n$. If $r > n$ then $S$ is linearly dependent.

This says that a linearly independent set in $\mathbb{R}^n$ contains at most $n$ vectors.

eg. $\{(0,1), (2,-1), (1,3)\}$ is linearly dependent in $\mathbb{R}^2$.

note: $7(0,1) + 1(2,-1) - 2(1,3) = (0,0)$.

Our first methods of solving a linear system involved reduction of the coefficient matrix to (reduced) row echelon form. The next example demonstrates a general principle about matrices in ref and rref:

If an (augmented) matrix is in ref (rref) then the set of nonzero rows is linearly independent.

**Example 7.** Let $A = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 1 & a_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and let $\vec{r}_1 = (1, a_{12}, a_{13}, a_{14})$, $\vec{r}_2 = (0,0,1,a_{24})$, $\vec{r}_3 = (0,0,0,1)$.

Show that the equation $c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = \vec{0}$ has only the trivial solution $c_1 = c_2 = c_3 = 0$.

$$c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = \vec{0} \Rightarrow c_1(1, a_{12}, a_{13}, a_{14}) + c_2(0,0,1,a_{24}) + c_3(0,0,0,1) = \vec{0}$$

$$\Rightarrow \begin{cases} c_1 = 0 \\ c_1 a_{12} = 0 \\ c_1 a_{13} + c_2 = 0 \\ c_1 a_{14} + c_2 a_{24} + c_3 = 0 \end{cases} \Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0$$

Thus $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are linearly independent.

Given functions $f_1(x), f_2(x), \ldots, f_n(x)$ that are differentiable $n-1$ times on $(-\infty, \infty)$, the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

"differentiate $n-1$ times"

is the Wronskian of $f_1, f_2, \ldots, f_n$.

**Theorem.** If the Wronskian of the functions $f_1, f_2, \ldots, f_n$ is not identically zero on $(-\infty, \infty)$, then the functions are linearly independent.

note: The converse is not true!!!
That is, if $W(x) = 0$ for all $x$, then $f_1, \ldots, f_n$ could be either linearly independent or linearly dependent.

**Example 8.** Show that $f(x) = x$ and $g(x) = \cos x$ are linearly independent in $C^\infty(-\infty, \infty)$.

$$W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x\sin x - \cos x.$$

Because $W(x)$ is not identically zero (eg. $W(0) = -1$), $f(x)$ and $g(x)$ are linearly independent.

**Example 9.** Show that $f_1(x) = 1$, $f_2(x) = e^x$, $f_3(x) = e^{2x}$ are linearly independent in $C^\infty(-\infty, \infty)$.

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} = 4e^{3x} - 2e^{3x} = 2e^{3x}.$$

Because $W(x)$ is not identically zero (eg. $W(0) = 2$), the functions $f_1, f_2, f_3$ are linearly independent in $C^\infty(-\infty, \infty)$.

## Section 4.5 Coordinates and Basis

**Objectives.**

- Introduce the idea of a basis for a vector space.
- Find coordinates for a vector relative to a given basis.

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A vector space $V$ is finite-dimensional if there is a finite set of vectors $S$ that spans $V$. Otherwise, $V$ is infinite-dimensional.

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors in a finite-dimensional vector space $V$. We say that $S$ is a basis for $V$ if the following two conditions hold.

- $S$ spans $V$ i.e. every vector in $V$ is a linear combination of vectors in $S$.
- $S$ is linearly independent i.e. if $k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n = \vec{0}$ then $k_1 = k_2 = \cdots = k_n = 0$.

**Example 1.** The set $\boxed{S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}}$ is a basis for $\mathbb{R}^n$. "standard basis for $\mathbb{R}^n$"

From Example 1, Section 4.3, $\mathbb{R}^n = \text{span}(S)$.

From Example 1, Section 4.4, $S$ is linearly independent.

Therefore, $S$ is a basis for $\mathbb{R}^n$.

**Example 2.** The set $\boxed{S = \{1, x, x^2, \ldots, x^n\}}$ is a basis for $P_n$. "standard basis for $P_n$"

From Example 3, Section 4.3, $P_n = \text{span}(S)$.

From Example 4, Section 4.4, $S$ is linearly independent.

Therefore, $S$ is a basis for $P_n$.

**Example 3.** The vector space $P_\infty$ is infinite-dimensional.

If $S = \{p_1, p_2, \cdots, p_r\}$ is a finite set of polynomials, then ~~there~~ $S$ contains a polynomial of maximum degree, say degree $n$. Then any linear combination of polynomials in $S$ has degree at most $n$. Thus we cannot express $x^{n+1}$ as a linear combination of polynomials in $S$, so $S$ does not span $P_\infty$.

Therefore, $P_\infty$ is infinite-dimensional.



note: $F(-\infty, \infty)$ is also infinite-dimensional.

**Example 4.** Show that the vectors $\vec{v}_1 = (1, 2, 1)$, $\vec{v}_2 = (2, 9, 0)$, $\vec{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

- linear independence:

If $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, then

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 \qquad + 4c_3 &= 0. \end{aligned}$$

- spanning set:

If $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = (b_1, b_2, b_3)$, then

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 \qquad + 4c_3 &= b_3. \end{aligned}$$

Because $\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} = -1 \neq 0$, the homogeneous system has only the trivial solution (so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent), and the nonhomogeneous is consistent for all vectors $(b_1, b_2, b_3)$ in $\mathbb{R}^3$.

Therefore, $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

**Example 5.** Show that the matrices $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis for the vector space $M_{22}$.

- linear independence:

If $c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, so $M_1, M_2, M_3, M_4$ are linearly independent.

- spanning set:

If $c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Taking $c_1 = a$, $c_2 = b$, $c_3 = c$, $c_4 = d$ satisfies this equation, so $M_1, M_2, M_3, M_4$ span $M_{22}$.

Therefore, $\boxed{S = \{M_1, M_2, M_3, M_4\}}$ is a basis for $M_{22}$.

"standard basis for $M_{22}$"

**Theorem.** Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for the vector space $V$. Then every vector $\vec{v}$ in $V$ can be written as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

in exactly one way.

**Proof.** Because $S$ is a basis for $V$, every vector in $V$ can be written as a linear combination of vectors in $S$.

Suppose $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ and $\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$.

Then $\vec{0} = (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \cdots + (c_n - d_n)\vec{v}_n$.

Because $S$ is linearly independent, we have $c_1 - d_1 = 0$, $c_2 - d_2 = 0$, $\cdots$, $c_n - d_n = 0$.

Thus $c_1 = d_1$, $c_2 = d_2$, $\cdots$, $c_n = d_n$.

Therefore, $\vec{v}$ ~~be~~ can be written as a linear combination of the basis $S$ in exactly one way.

The numbers $c_1, c_2, \ldots, c_n$ in this theorem are called the coordinates of $\vec{v}$ relative to the basis $S$. The vector $(c_1, c_2, \ldots, c_n)$ is called the coordinate vector of $\vec{v}$ relative to the basis $S$, and is denoted by

$$(\vec{v})_S = (c_1, c_2, \ldots, c_n).$$

If $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$, then

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \iff (\vec{v})_S = (c_1, c_2, \cdots, c_n).$$

**Example 6.** Consider the standard basis $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for $\mathbb{R}^3$. What is the coordinate vector for $\vec{v} = (a, b, c)$ relative to the basis $S$?

$\vec{v} = (a, b, c) = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$, so $(\vec{v})_S = (a, b, c)$.

**Example 7.** Consider the basis $S = \{(1, 0), (1, 2)\}$ for $\mathbb{R}^2$. What is the coordinate vector for $\vec{v} = (-1, 4)$ relative to the basis $S$?

$\vec{v} = (-1, 4) = -3(1, 0) + 2(1, 2)$, so $(\vec{v})_S = (-3, 2)$.

**Example 8.** Find the coordinate vector for the polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ relative to the standard basis for $P_n$.

$$p(x) = a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n),$$

$$\text{so} \quad (p(x))_S = (a_0, a_1, a_2, \ldots, a_n).$$

**Example 9.** Find the coordinate vector for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ relative to the standard basis for $M_{22}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\text{so} \quad (A)_S = (a, b, c, d).$$

**Example 10.** Recall from Example 4 that $\vec{v}_1 = (1, 2, 1)$, $\vec{v}_2 = (2, 9, 0)$, $\vec{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

(a) Find the coordinate vector for $\vec{v} = (5, -1, 9)$ relative to the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

From $(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$, we obtain

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 5 \\ 2c_1 + 9c_2 + 3c_3 = -1 \\ c_1 \phantom{+ 9c_2} + 4c_3 = 9 \end{cases}.$$

The solution is $c_1 = 1$, $c_2 = -1$, $c_3 = 2$.

Therefore, $(\vec{v})_S = (1, -1, 2)$.

(b) Find $\vec{w}$ given that $(\vec{w})_S = (-1, 3, 2)$.

$$\vec{w} = -1\vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_3 = -1(1,2,1) + 3(2,9,0) + 2(3,3,4)$$

$$= \underline{(11, 31, 7)}.$$

## Section 4.6 Dimension

**Objectives.**

- Define the dimension of a finite-dimensional vector space.
- Relate dimension to span and linear independence.

**Theorem.** Every basis for a finite-dimensional vector space $V$ contains the same number of vectors.

The number of vectors in a basis for the finite-dimensional vector space $V$ is called the dimension of $V$, and is denoted by $\dim V$.

note: if $V = \{\vec{0}\}$, then $\dim V = 0$.

**Example 1.** What is the dimension of each vector space?

(a) $\mathbb{R}^n$ The standard basis is $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$.

Thus $\dim(\mathbb{R}^n) = n$.

(b) $P_n$ The standard basis is $\{1, x, \dots, x^n\}$.

Thus $\dim(P_n) = n+1$.

(c) $M_{mn}$

$\dim(M_{mn}) = mn$. e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in the standard basis for $M_{23}$.

**Theorem.** Let $V$ be a finite-dimensional vector space with $\dim V = n$.

1. If $W$ is a subset of $V$ that contains more than $n$ vectors, then $W$ is linearly dependent.
2. If $W$ is a subset of $V$ that contains fewer than $n$ vectors, then $W$ does not span $V$.

**Example 2.** Suppose that $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ is a linearly independent set of vectors in a vector space $V$. What is $\dim(\operatorname{span}(S))$? Why?

$S$ is linearly independent, and $S$ spans $\operatorname{span}(S)$. This means that $S$ is a basis for $\operatorname{span}(S)$, so $\dim(\operatorname{span}(S)) = r$.

**Example 3.** Consider the linear system below. (This is Example 5 from the Section 1.2 lecture notes.)

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0\\
5x_3 + 10x_4 \qquad + 15x_6 &= 0\\
2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

The general solution to this system is

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0.$$

(a) Write the solution in vector form.

$$\vec{x} = (-3r-4s-2t,\ r,\ -2s,\ s,\ t,\ 0)$$
$$= r(-3,1,0,0,0,0) + s(-4,0,-2,1,0,0) + t(-2,0,0,0,1,0).$$

these vectors are linearly independent!!!

(b) Find a basis for the solution space of the system.

Every vector $\vec{x}$ in the solution space is a linear combination of $(-3,1,0,0,0,0)$, $(-4,0,-2,1,0,0)$, $(-2,0,0,0,1,0)$, and these vectors are linearly independent. Thus $\{(-3,1,0,0,0,0), (-4,0,-2,1,0,0), (-2,0,0,0,1,0)\}$ is a basis for the solution space.

(c) What is the dimension of the solution space?

There are three vectors in <u>any</u> basis, so this space has dimension 3.

"union" (i.e. add $\vec{v}$ to the set $S$).

**Theorem.** Let $S$ be a set of vectors in a vector space $V$.

1. If $S$ is linearly independent, and $\vec{v}$ is not in $\text{span}(S)$, then $S \cup \{\vec{v}\}$ is linearly independent.
   i.e. adding a vector outside span(S) does not affect linear independence.
2. If $\vec{v}$ is in $S$, and $\vec{v}$ can be written as a (nonzero) linear combination of other vectors in $S$, then

remove $\vec{v}$ from $S$.

$$\text{span}(S) = \text{span}(S - \{\vec{v}\}).$$

i.e. removing linearly dependent vectors does not affect span(S).

**Example 4.** Explain why the polynomials $p(x) = 1 + x^2$, $q(x) = 2 + x^2$, $r(x) = x^3$ are linearly independent.

$p(x)$ and $q(x)$ are linearly independent (neither is a multiple of the other). Also, $r(x)$ is not in $\text{span}\{p(x), q(x)\}$, because $r$ is cubic but $p, q$ are quadratic. Thus $\{p(x), q(x), r(x)\}$ is linearly independent.

**Theorem.** Let $V$ be a vector space with $\dim V = n$, and let $S$ be a set of $n$ vectors in $V$.

equal!!!

1. $S$ is a basis for $V$ if and only if $S$ is linearly independent.
2. $S$ is a basis for $V$ if and only if $S$ spans $V$.

**Example 5.** Explain why each set of vectors is a basis for the given vector space.

(a) $\vec{v}_1 = (1, 4)$ and $\vec{v}_2 = (3, -2)$ in $\mathbb{R}^2$

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent, and $\dim(\mathbb{R}^2) = 2$.

Thus $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$.

(b) $\vec{v}_1 = (1, 0, 2)$, $\vec{v}_2 = (-1, 0, 1)$, and $\vec{v}_3 = (2, -2, 3)$ in $\mathbb{R}^3$

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent in the $xz$-plane.

because the $y$-coord. is $\neq 0$.

Because $\vec{v}_3$ is not in the $xz$-plane (i.e. $\vec{v}_3$ is not in $\text{span}\{\vec{v}_1, \vec{v}_2\}$), the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Also, $\dim(\mathbb{R}^3) = 3$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

**Theorem.** Let $V$ be a vector space with $\dim V = n$, and let $S$ be a set of vectors in $V$.

1. If $S$ spans $V$ but is not a basis for $V$, then $S$ can be reduced to a basis for $V$ by removing some vectors.
2. If $S$ is linearly independent but is not a basis for $V$, then $S$ can be enlarged to a basis for $V$ by adding some vectors.

**Example 6.** (a) Find a subset of $S = \{(1,-1), (-1,1), (1,1)\}$ that is a basis for $\mathbb{R}^2$.

$\dim(\mathbb{R}^2) = 2$, so we need two vectors from $S$ to form a basis.

The vectors $(1,-1)$ and $(1,1)$ are linearly independent.

Thus $\{(1,-1),(1,1)\}$ is a basis for $\mathbb{R}^2$.

note: $\{(-1,1),(1,1)\}$ is also a basis for $\mathbb{R}^2$, but

$\{(1,-1),(-1,1)\}$ is not a basis for $\mathbb{R}^2$. (why?)

(b) Enlarge the set $S = \{(1,1,0), (1,0,-1)\}$ to a basis for $\mathbb{R}^3$.

Let's try adding $(1,0,0)$ to $S$.

$k_1(1,1,0) + k_2(1,0,-1) + k_3(1,0,0) = (0,0,0)$

$\Rightarrow (k_1 + k_2 + k_3, k_1, -k_2) = (0,0,0)$

$\Rightarrow k_1 = k_2 = k_3 = 0$

Thus $\{(1,1,0),(1,0,-1),(1,0,0)\}$ is linearly independent and contains three vectors, so this is a basis for $\mathbb{R}^3$.

note: $\{(1,1,0),(1,0,-1),(0,1,0)\}$ and $\{(1,1,0),(1,0,-1),(0,0,1)\}$ are also bases for $\mathbb{R}^3$.

**Theorem.** If $W$ is a subspace of a finite-dimensional vector space $V$, then:

1. $W$ is finite-dimensional.
2. $\dim W \leq \dim V$.
3. $W = V$ if and only if $\dim W = \dim V$.

## Section 4.7 Change of Basis

**Objectives.**

- Introduce the 'change of basis problem'.
- Define the transition matrix for a change of basis.
- Find the transition matrix for a change of basis.

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Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for a vector space $V$, and let $\vec{v}$ be a vector in $V$. Recall the definition of the coordinate vector for $\vec{v}$ relative to $B$:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \iff [\vec{v}]_B = (c_1, c_2, \cdots, c_n) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

The set of all such coordinate vectors for $V$ is a function from $V$ to $\mathbb{R}^n$ called the coordinate map relative to $B$.

i.e. for each vector $\vec{v}$ in $V$ and each basis $B$ for $V$, there is a coordinate vector $[\vec{v}]_B$ in $\mathbb{R}^n$.

Sometimes we may want to change from one basis $B$ for $V$ to a different basis $B'$. Thus we would like to know how $[\vec{v}]_B$ and $[\vec{v}]_{B'}$ are related.

Suppose that $B = \{\vec{u}_1, \vec{u}_2\}$ and $B' = \{\vec{u}_1', \vec{u}_2'\}$ are both bases for $V$, and that $\vec{v}$ is a vector in $V$.

Let $[\vec{u}_1]_{B'} = \begin{bmatrix} a \\ b \end{bmatrix}$, $[\vec{u}_2]_{B'} = \begin{bmatrix} c \\ d \end{bmatrix}$, and $[\vec{v}]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

Then: $\vec{u}_1 = a\vec{u}_1' + b\vec{u}_2'$ and $\vec{u}_2 = c\vec{u}_1' + d\vec{u}_2'$, so

$$\vec{v} = k_1\vec{u}_1 + k_2\vec{u}_2 = k_1(a\vec{u}_1' + b\vec{u}_2') + k_2(c\vec{u}_1' + d\vec{u}_2') = (k_1a + k_2c)\vec{u}_1' + (k_1b + k_2d)\vec{u}_2'$$

Thus: $$[\vec{v}]_{B'} = \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{\text{transition matrix from } B \text{ to } B'}[\vec{v}]_B.$$

$$P_{B\to B'} = \left[ [\vec{u}_1]_{B'} \;\; [\vec{u}_2]_{B'} \right].$$

**Change of Basis Problem.** Suppose that $B = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is the old basis for $V$, and $B' = \{\vec{u}'_1, \vec{u}'_2, \dots, \vec{u}'_n\}$ is the new basis for $V$. Then the coordinate vectors for a vector $\vec{v}$ in $V$ satisfy

$$[\vec{v}]_{B'} = P_{B\to B'}\,[\vec{v}]_B$$

where $P_{B\to B'} = \big[[\vec{u}_1]_{B'} \quad [\vec{u}_2]_{B'} \quad \cdots \quad [\vec{u}_n]_{B'}\big]$ is the transition matrix from $B$ to $B'$.

The columns of the transition matrix are *the coordinate vectors of the old basis relative to the new basis.*

**Example 1.** Consider the bases $B = \{(1,0),(0,1)\}$ and $B' = \{(1,1),(1,2)\}$ for $\mathbb{R}^2$.

(a) Find the transition matrix $P_{B\to B'}$ from $B$ to $B'$.

$\vec{u}_1 = (1,0) = 2(1,1) - (1,2) = 2\vec{u}_1' - \vec{u}_2'$, so $[\vec{u}_1]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$\vec{u}_2 = (0,1) = -(1,1) + (1,2) = -\vec{u}_1' + \vec{u}_2'$, so $[\vec{u}_2]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Thus: $P_{B\to B'} = \big[[\vec{u}_1]_{B'} \ \ [\vec{u}_2]_{B'}\big] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

(b) Find the transition matrix $P_{B'\to B}$ from $B'$ to $B$.

$\vec{u}_1' = (1,1) = (1,0) + (0,1) = \vec{u}_1 + \vec{u}_2$, so $[\vec{u}_1']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\vec{u}_2' = (1,2) = \cdots = \vec{u}_1 + 2\vec{u}_2$, so $[\vec{u}_2']_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Thus: $P_{B'\to B} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(c) Suppose that $[\vec{v}]_B = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Find $[\vec{v}]_{B'}$.

$$[\vec{v}]_{B'} = P_{B\to B'}[\vec{v}]_B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}.$$

Applying a change of basis from $B$ to $B'$ and then a change of basis from $B'$ to $B$ leaves coordinate vectors unchanged.

i.e. $$[\vec{v}]_B = P_{B'\to B}[\vec{v}]_{B'} = P_{B'\to B}\left(P_{B\to B'}[\vec{v}]_B\right)$$
$$= \left(P_{B'\to B}P_{B\to B'}\right)[\vec{v}]_B = I[\vec{v}]_B,$$

so $$P_{B'\to B}P_{B\to B'} = I.$$

This means that the transition matrices $P_{B\to B'}$ and $P_{B'\to B}$ are inverses of each other.

**Theorem.** If $P$ is the transition matrix from a basis $B$ to a basis $B'$ in the vector space $V$, then $P$ is invertible and $P^{-1}$ is the transition matrix from $B'$ to $B$.

We can find a transition matrix by row-reducing the matrix that has the vectors from each basis as columns.

$$\left[\vec{u}_1' \;\; \vec{u}_2' \,\middle|\, \vec{u}_1 \;\; \vec{u}_2\right] \xrightarrow{\text{row operations}} \left[I \,\middle|\, P_{B\to B'}\right]$$

$$\left[\vec{u}_1' \;\; \vec{u}_2' \,\middle|\, \vec{u}_1 \;\; \vec{u}_2\right] \xrightarrow{\text{row operations}} \left[P_{B'\to B} \,\middle|\, I\right]$$

new basis $B'$ old basis $B$

**Example 2.** Find the transition matrix $P_{B\to B'}$ for the bases in Example 1.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_2\to R_2-R_1]{} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_1\to R_1-R_2]{} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Thus $$P_{B\to B'} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

**Theorem.** Let $B = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be any basis for $\mathbb{R}^n$ and let $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then the transition matrix from $B$ to $S$ is

$$P_{B\to S} = [\vec{u}_1|\vec{u}_2|\cdots|\vec{u}_n].$$

In particular, if $A = [\vec{v}_1|\vec{v}_2|\cdots|\vec{v}_n]$ is an invertible matrix, then $A$ is a transition matrix from the basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ to the standard basis for $\mathbb{R}^n$.