Section 4.7 Change of Basis

Objectives.

- Introduce the 'change of basis problem'.
- Define the transition matrix for a change of basis.
- Find the transition matrix for a change of basis.

Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for a vector space V, and let $\vec{v}$ be a vector in V. Recall the definition of the coordinate vector for $\vec{v}$ relative to B:

The set of all such coordinate vectors for V is a function from V to $\mathbb{R}^n$ called the coordinate map relative to B.

Sometimes we may want to change from one basis B for V to a different basis B'. Thus we would like to know how $[\vec{v}]_B$ and $[\vec{v}]_{B'}$ are related.

Suppose that $B = \{\vec{u}_1, \vec{u}_2\}$ and $B' = \{\vec{u}'_1, \vec{u}'_2\}$ are both bases for V, and that $\vec{v}$ is a vector in V.

<u>Change of Basis Problem.</u> Suppose that $B = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is the old basis for V, and $B' = \{\vec{u}'_1, \vec{u}'_2, \ldots, \vec{u}'_n\}$ is the new basis for V. Then the coordinate vectors for a vector $\vec{v}$ in V satisfy

$$[\vec{v}]_{B'} = P_{B \to B'} [\vec{v}]_B$$

where $P_{B \to B'} = \begin{bmatrix} [\vec{u}_1]_{B'} & [\vec{u}_2]_{B'} & \cdots & [\vec{u}_n]_{B'} \end{bmatrix}$ is the <u>transition matrix from B to B'</u>.

The columns of the transition matrix are *the coordinate vectors of the old basis relative to the new basis*.

Example 1. Consider the bases $B = \{(1, 0), (0, 1)\}$ and $B' = \{(1, 1), (1, 2)\}$ for $\mathbb{R}^2$.

(a) Find the transition matrix $P_{B \to B'}$ from B to B'.

(b) Find the transition matrix $P_{B' \to B}$ from B' to B.

(c) Suppose that $[\vec{v}]_B = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Find $[\vec{v}]_{B'}$.

Applying a change of basis from B to B' and then a change of basis from B' to B leaves coordinate vectors unchanged.

This means that the transition matrices $P_{B \to B'}$ and $P_{B' \to B}$ are inverses of each other.

Theorem. If P is the transition matrix from a basis B to a basis B' in the vector space V, then P is invertible and P^{-1} is the transition matrix from B' to B.

We can find a transition matrix by row-reducing the matrix that has the vectors from each basis as columns.

Example 2. Find the transition matrix $P_{B \to B'}$ for the bases in Example 1.

Theorem. Let $B = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be any basis for $\mathbb{R}^n$ and let $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then the transition matrix from B to S is

$$P_{B \to S} = \left[\vec{u}_1 | \vec{u}_2 | \cdots | \vec{u}_n\right].$$

In particular, if $A = \left[\vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_n\right]$ is an invertible matrix, then A is a transition matrix from the basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ to the standard basis for $\mathbb{R}^n$.