Section 4.6 Dimension

Objectives.

- Define the dimension of a finite-dimensional vector space.
- Relate dimension to span and linear independence.

Theorem. Every basis for a finite-dimensional vector space V contains the same number of vectors.

The number of vectors in a basis for the finite-dimensional vector space V is called the dimension of V, and is denoted by $\dim V$.

Example 1. What is the dimension of each vector space?

(a) $\mathbb{R}^n$

(b) P_n

(c) M_{mn}

Theorem. Let V be a finite-dimensional vector space with $\dim V = n$.

1. If W is a subset of V that contains more than n vectors, then W is linearly dependent.

2. If W is a subset of V that contains fewer than n vectors, then W does not span V.

Example 2. Suppose that $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ is a linearly independent set of vectors in a vector space V. What is $\dim(\text{span}(S))$? Why?

Example 3. Consider the linear system below. (This is Example 5 from the Section 1.2 lecture notes.)

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \phantom{{}- 2x_4} + 2x_5 \phantom{{}- 3x_6} &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 \phantom{{}+ 4x_5} + 15x_6 &= 0 \\ 2x_1 + 6x_2 \phantom{{}- 5x_3} + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

The general solution to this system is

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0.$$

(a) Write the solution in vector form.

(b) Find a basis for the solution space of the system.

(c) What is the dimension of the solution space?

Theorem. Let S be a set of vectors in a vector space V.

1. If S is linearly independent, and $\vec{v}$ is not in $\text{span}(S)$, then $S \cup \{\vec{v}\}$ is linearly independent.

2. If $\vec{v}$ is in S, and $\vec{v}$ can be written as a (nonzero) linear combination of other vectors in S, then

$$\text{span}(S) = \text{span}(S - \{\vec{v}\}).$$

Example 4. Explain why the polynomials $p(x) = 1 + x^2$, $q(x) = 2 + x^2$, $r(x) = x^3$ are linearly independent.

Theorem. Let V be a vector space with $\dim V = n$, and let S be a set of n vectors in V.

1. S is a basis for V if and only if S is linearly independent.

2. S is a basis for V if and only if S spans V.

Example 5. Explain why each set of vectors is a basis for the given vector space.

(a) $\vec{v}_1 = (1, 4)$ and $\vec{v}_2 = (3, -2)$ in $\mathbb{R}^2$

(b) $\vec{v}_1 = (1, 0, 2)$, $\vec{v}_2 = (-1, 0, 1)$, and $\vec{v}_3 = (2, -2, 3)$ in $\mathbb{R}^3$

Theorem. Let V be a vector space with $\dim V = n$, and let S be a set of vectors in V.

1. If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing some vectors.

2. If S is linearly independent but is not a basis for V, then S can be enlarged to a basis for V by adding some vectors.

Example 6. (a) Find a subset of $S = \{(1, -1), (-1, 1), (1, 1)\}$ that is a basis for $\mathbb{R}^2$.

(b) Enlarge the set $S = \{(1, 1, 0), (1, 0, -1)\}$ to a basis for $\mathbb{R}^3$.

Theorem. If W is a subspace of a finite-dimensional vector space V, then:

1. W is finite-dimensional.

2. $\dim W \leq \dim V$.

3. $W = V$ if and only if $\dim W = \dim V$.