**Section 4.5 Coordinates and Basis**

**Objectives.**

- Introduce the idea of a basis for a vector space.
- Find coordinates for a vector relative to a given basis.

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A vector space $V$ is finite-dimensional if there is a finite set of vectors $S$ that spans $V$. Otherwise, $V$ is infinite-dimensional.

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors in a finite-dimensional vector space $V$. We say that $S$ is a basis for $V$ if the following two conditions hold.

- $S$ spans $V$
- $S$ is linearly independent

**Example 1.** The set $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$.

**Example 2.** The set $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n$.

**Example 3.** The vector space $P_\infty$ is infinite-dimensional.

**Example 4.** Show that the vectors $\vec{v}_1 = (1, 2, 1)$, $\vec{v}_2 = (2, 9, 0)$, $\vec{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

**Example 5.** Show that the matrices $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis for the vector space $M_{22}$.

**Theorem.** Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for the vector space $V$. Then every vector $\vec{v}$ in $V$ can be written as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

in exactly one way.

**Proof.**

The numbers $c_1, c_2, \ldots, c_n$ in this theorem are called the coordinates of $\vec{v}$ relative to the basis $S$. The vector $(c_1, c_2, \ldots, c_n)$ is called the coordinate vector of $\vec{v}$ relative to the basis $S$, and is denoted by

$$(\vec{v})_S = (c_1, c_2, \ldots, c_n).$$

**Example 6.** Consider the standard basis $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for $\mathbb{R}^3$. What is the coordinate vector for $\vec{v} = (a, b, c)$ relative to the basis $S$?

**Example 7.** Consider the basis $S = \{(1, 0), (1, 2)\}$ for $\mathbb{R}^2$. What is the coordinate vector for $\vec{v} = (-1, 4)$ relative to the basis $S$?

**Example 8.** Find the coordinate vector for the polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ relative to the standard basis for $P_n$.

**Example 9.** Find the coordinate vector for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ relative to the standard basis for $M_{22}$.

**Example 10.** Recall from Example 4 that $\vec{v}_1 = (1, 2, 1)$, $\vec{v}_2 = (2, 9, 0)$, $\vec{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

(a) Find the coordinate vector for $\vec{v} = (5, -1, 9)$ relative to the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(b) Find $\vec{w}$ given that $(\vec{w})_S = (-1, 3, 2)$.