Section 4.4 Linear Independence

Objectives.

- Define linear independence of vectors.
- Determine whether a set of vectors is linearly independent or linearly dependent.
- Define and apply the Wronskian to determine whether a set of functions is linearly independent.

Let V be a vector space. A nonempty set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in V is linearly independent if no vector in S can be written as a linear combination of the other vectors in S. Otherwise, S is linearly dependent.

Note: If $S = \{\vec{v}\}$ contains one vector, then S is linearly independent if $\vec{v} \neq \vec{0}$ and linearly dependent if $\vec{v} = \vec{0}$.

Theorem. A nonempty set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in V is linearly independent if and only if the only solution to the equation

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \vec{0}$$

is $k_1 = k_2 = \cdots = k_r = 0$.

Example 1. The set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of standard unit vectors in $\mathbb{R}^n$ is linearly independent.

Example 2. Determine whether the vectors $\vec{v}_1 = (1, -2, 3)$, $\vec{v}_2 = (5, 6, -1)$, $\vec{v}_3 = (3, 2, 1)$ are linearly independent in $\mathbb{R}^3$.

Example 3. Determine whether the vectors $\vec{v}_1 = (1, 2, 2, -1)$, $\vec{v}_2 = (4, 9, 9, -4)$, $\vec{v}_3 = (5, 8, 9, -5)$ are linearly independent in $\mathbb{R}^4$.

Example 4. The set $\{1, x, x^2, \ldots, x^n\}$ of polynomials in P_n is linearly independent.

Example 5. Determine whether the polynomials $p_1(x) = 1 - x$, $p_2(x) = 5 + 3x - 2x^2$, $p_3(x) = 1 + 3x - x^2$ are linearly independent in P_2.

Theorem. Let S be a nonempty set of vectors in a vector space V.

(a) If $\vec{0}$ is in S then S is linearly dependent.

(b) If S contains exactly two vectors, then S is linearly independent if and only if neither vector is a scalar multiple of the other.

Example 6. Recall that $F(-\infty, \infty)$ is the set of all functions defined on $(-\infty, \infty)$.

(a) Show that the functions $f(x) = x$ and $g(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$

(b) Show that the functions $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$ are linearly dependent in $F(-\infty, \infty)$

The second condition in the previous theorem can be interpreted – and extended – geometrically as follows.

- Two distinct nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are linearly dependent if and only if they are parallel – that is, they lie on the same line.
- Three distinct nonzero vectors in $\mathbb{R}^3$ are linearly dependent if and only if they lie in the same plane.

<u>Theorem.</u> Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a nonempty set of vectors vectors in $\mathbb{R}^n$. If $r > n$ then S is linearly dependent.

Our first methods of solving a linear system involved reduction of the coefficient matrix to (reduced) row echelon form. The next example demonstrates a general principle about matrices in ref and rref:

Example 7. Let $A = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 1 & a_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and let $\vec{r}_1 = (1, a_{12}, a_{13}, a_{14})$, $\vec{r}_2 = (0, 0, 1, a_{24})$, $\vec{r}_3 = (0, 0, 0, 1)$.

Show that the equation $c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = \vec{0}$ has only the trivial solution $c_1 = c_2 = c_3 = 0$.

Given functions $f_1(x), f_2(x), \dots, f_n(x)$ that are differentiable $n-1$ times on $(-\infty, \infty)$, the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is the Wronskian of $f_1, f_2, \dots, f_n$.

Theorem. If the Wronskian of the functions $f_1, f_2, \dots, f_n$ is not identically zero on $(-\infty, \infty)$, then the functions are linearly independent.

Example 8. Show that $f(x) = x$ and $g(x) = \cos x$ are linearly independent in $C^\infty(-\infty, \infty)$.

Example 9. Show that $f_1(x) = 1$, $f_2(x) = e^x$, $f_3(x) = e^{2x}$ are linearly independent in $C^\infty(-\infty, \infty)$.