Section 4.3 Spanning Sets

Objectives.

- Introduce the span of a set of vectors.
- Define spanning sets for a subspace of a vector space.
- Discuss examples of spanning sets in real vector spaces.

Let V be a vector space, and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ be vectors in V. The vector $\vec{w}$ in V is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ if there are scalars $k_1, k_2, \ldots, k_r$ such that

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r.$$

Theorem. If $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_r\}$ is a nonempty set of vectors in a vector space V, then:

(a) The set W of all linear combinations of vectors in S is a subspace of V.

(b) The set W in part (a) is the smallest subspace of V that contains all the vectors in S.
(This means that any other subspace of V that contains S also contains every vector in W.)

Proof of (a).

Proof of (b).

The subspace W in this theorem is the called subspace of V spanned by S, and we say that the vectors in S span the subspace W.

Example 1. Every vector in $\mathbb{R}^n$ can be written as a linear combination of the vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$.

Example 2. (a) Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. Describe $\mathrm{span}\{\vec{v}\}$.

(b) Let $\vec{v}_1$ and $\vec{v}_2$ be non-parallel vectors in $\mathbb{R}^3$. Describe $\mathrm{span}\{\vec{v}_1, \vec{v}_2\}$.

Example 3. Every polynomial in P_n can be written as a linear combination of the polynomials $1, x, x^2, \dots, x^n$.

There are two important problems we can ask about spanning sets in a vector space.

- Given a set of vectors S and a vector $\vec{v}$, decide whether $\vec{v}$ is in $\mathrm{span}(S)$.
- Given a set of vectors S, decide whether $\mathrm{span}(S) = V$.

Example 4. Let $\vec{u} = (1, 2, -1)$ and $\vec{v} = (6, 4, 2)$.

(a) Show that $\vec{w_1} = (9, 2, 7)$ is a linear combination of $\vec{u}$ and $\vec{v}$.

(b) Show that $\vec{w_2} = (4, -1, 8)$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

Example 5. Determine whether the vectors $\vec{v}_1 = (1, 1, 2)$, $\vec{v}_2 = (1, 0, 1)$, and $\vec{v}_3 = (2, 1, 3)$ span $\mathbb{R}^3$.

Strategy. To determine whether the set $S = \{\vec{w}_1, \ldots, \vec{w}_r\}$ spans the vector space V:

Example 6. Determine whether the set S spans P_2.

(a) $S = \{1 + x + x^2, -1 - x, 2 + 2x + x^2\}$

(b) $S = \{x + x^2, x - x^2, 1 + x, 1 - x\}$

Example 7. Determine whether the set S spans M_{22}.

(a) $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

(b) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$