**Section 4.2 Subspaces**

**Objectives.**

- Introduce the notion of a subspace of a vector space.
- Determine whether a subset of a vector space is a subspace.
- Discuss some subspaces of real vector spaces.

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Recall that a vector space is a set $V$ that generalizes the vector arithmetic of $\mathbb{R}^n$ – vectors in $V$ can be added or scaled without leaving $V$, and these operations are consistent with the usual rules of arithmetic.

Suppose that $W$ is a set of vectors in a vector space $V$. We call $W$ a subspace of $V$ if $W$ is a vector space with the operations of addition and scalar multiplication from $V$.

**Example 1.** If $V$ is any vector space, and $\vec{0}$ is the zero vector in $V$, then $W = \left\{\vec{0}\right\}$ is a subspace of $V$.

Six of the ten axioms for a vector space are satisfied by every subset of vectors. The four axioms that need to be checked are:

**Subspace Test.** If $W$ is a nonempty set of vectors in a vector space $V$, then $W$ is a subspace of $V$ if and only if both of the following conditions are satisfied.

1. If $\vec{u}$ and $\vec{v}$ are in $W$, then $\vec{u} + \vec{v}$ is in $W$.

2. If $\vec{u}$ is in $V$ and $k$ is a scalar, then $k\vec{u}$ is in $V$.

**Strategy.** To show that $W$ is a subspace of $V$:

**Example 2.** If $W$ is a line through the origin in $\mathbb{R}^n$, then $W$ is a subspace of $\mathbb{R}^n$.

**Example 3.** If $W$ is a plane through the origin in $\mathbb{R}^3$, then $W$ is a subspace of $\mathbb{R}^3$.

**Example 4.** The set $W$ of all points $(x, y)$ in $\mathbb{R}^2$ with $x \geq 0$ and $y \geq 0$ is not a subspace of $\mathbb{R}^2$.

| **Subspaces of $\mathbb{R}^2$.** | **Subspaces of $\mathbb{R}^3$.** |
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Recall that $M_{nn}$ is the vector space of all $n \times n$ matrices of real numbers.

**Example 5.** Let $W$ be the set of all symmetric $n \times n$ matrices.

(a) Discuss why $W$ is a subspace of $M_{nn}$.

(b) What are some other subspaces of $M_{nn}$?

**Example 6.** Let $W$ be the set of all invertible $2 \times 2$ matrices.

(a) Find two matrices $A$ and $B$ in $W$ such that $A + B$ is not in $W$. (What does this example show?)

(b) Find a matrix $A$ and a scalar $k$ such that $kA$ is not in $W$. (What does this example show?)

*Note: more generally, the set of all invertible $n \times n$ matrices is not a subspace of $M_{nn}$.*

Recall that $F(-\infty, \infty)$ is the set of all (real-valued) functions defined on the interval $(-\infty, \infty)$.

**Example 7.** The set $C(-\infty, \infty)$ of all *continuous* functions defined on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$.

**Example 8.** The following sets of functions are subspaces of $F(-\infty, \infty)$.

(a) $C^1(-\infty, \infty)$

(b) $C^m(-\infty, \infty)$

(c) $C^\infty(-\infty, \infty)$

A polynomial of degree $n$ is a function that can be written

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$.

**Example 9.** The set $P_\infty$ of all polynomials is a subspace of $F(-\infty, \infty)$.

**Example 10.** The set $P_n$ of all polynomials with degree at most $n$ is a subspace of $F(-\infty, \infty)$.

**Example 11.** Determine whether each set of matrices is a subspace of $M_{22}$.

(a) The set $U$ of all matrices of the form $\begin{bmatrix} x & 2x \\ 0 & y \end{bmatrix}$.

(b) The set $W$ of all $2 \times 2$ matrices $A$ such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

**Example 12.** Determine whether each set of polynomials is a subspace of $P_2$.

(a) The set $U$ of all polynomials of the form $p(x) = 1 - ax + ax^2$.

(b) The set $W$ of all polynomials such that $p(3) = 0$.

**<u>Theorem.</u>** If $W_1, W_2, \ldots, W_k$ are all subspaces of a vector space $V$, then the set $W$ of all vectors in the intersection of these subspaces is a subspace of $V$.

**<u>Theorem.</u>** Let $A$ be an $m \times n$ matrix. The set of all solutions $\vec{x}$ to the homogeneous linear system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

The <u>solution space</u> in the previous theorem is called the <u>kernel</u> of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

**<u>Theorem.</u>** Let $A$ be an $m \times n$ matrix. Then the kernel of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$.

**Example 13.** Describe the geometry of the solution space for each homogeneous linear system.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$