Section 4.1 Real Vector Spaces

Objectives.

- Introduce the vector space axioms.
- Discuss some examples of real vector spaces.

A vector space is a generalization of the vector arithmetic in $\mathbb{R}^n$. A (nonempty) set of objects forms a vector space if it satisfies ten assumptions (axioms) that describe the rules of arithmetic for two operations.

Vector space axioms. Let V be a (nonempty) set of objects with two operations called *addition* and *scalar multiplication*. If the following ten axioms are satisfied by all $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and all scalars k and m, then V is a vector space.

1. If $\vec{u}$ and $\vec{v}$ are in V, then $\vec{u}+\vec{v}$ is in V.
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
4. There exists a vector $\vec{0}$ in V that satisfies $\vec{0}+\vec{u}=\vec{u}+\vec{0}=\vec{u}$ for all $\vec{u}$ in V.
5. For each $\vec{u}$ in V, the vector $-\vec{u}$ (the negative of $\vec{u}$) is in V and satisfies $\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=\vec{0}$.
6. If $\vec{u}$ is in V and k is a scalar, then $k\vec{u}$ is in V.
7. $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
8. $(k+m)\vec{u}=k\vec{u}+m\vec{u}$
9. $k(m\vec{u})=(km)\vec{u}$
10. $1\vec{u}=\vec{u}$

Strategy. To show that a set V with two operations is a vector space:

Example 1. The set $V = \left\{\vec{0}\right\}$ with the operations

$$\vec{0} + \vec{0} = \vec{0} \qquad \text{and} \qquad k\vec{0} = \vec{0} \quad \text{for all scalars } k$$

is a vector space.

Example 2. The set $V = \mathbb{R}^n$ of all n-tuples of real numbers with the operations

$$\begin{aligned} \vec{u} + \vec{v} &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n), \\ k\vec{u} &= k(u_1, u_2, \ldots, u_n) = (ku_1, ku_2, \ldots, ku_n) \end{aligned}$$

is a vector space.

Example 3. The set $V = \mathbb{R}^\infty$ of all infinite sequences of real numbers with the operations

$$\begin{aligned} \vec{u} + \vec{v} &= (u_1, u_2, \ldots, u_n, \ldots) + (v_1, v_2, \ldots, v_n, \ldots) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots), \\ k\vec{u} &= k(u_1, u_2, \ldots, u_n, \ldots) = (ku_1, ku_2, \ldots, ku_n, \ldots) \end{aligned}$$

is a vector space.

Example 4. The set $V = M_{22}$ of all 2×2 matrices of real numbers with the operations

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}, \\ k\vec{u} &= k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \end{aligned}$$

is a vector space.

Example 5. The set $V = M_{mn}$ of all $m \times n$ matrices of real numbers

$$\vec{u} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \dots & u_{mn} \end{bmatrix}$$

with the operations of matrix addition and scalar multiplication is a vector space.

Example 6. Let $F(-\infty,\infty)$ be the set of all real-valued functions defined on the interval $(-\infty,\infty)$. For $\vec{f}=f(x)$ and $\vec{g}=g(x)$ in $F(-\infty,\infty)$, we define addition and scalar multiplication by

$$\vec{f}+\vec{g}=f(x)+g(x) \qquad \text{and} \qquad k\vec{f}=kf(x) \quad \text{for all scalars } k.$$

Then $V=F(-\infty,\infty)$ with these two operations is a vector space.

Example 7. Let $V=\mathbb{R}^2$. For $\vec{u}=(u_1,u_2)$ and $\vec{v}=(v_1,v_2)$ in $\mathbb{R}^2$, we define addition and scalar multiplication by

$$\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2) \qquad \text{and} \qquad k\vec{u}=(ku_1,0).$$

Then $V=\mathbb{R}^2$ with these two operations is **not** a vector space.

Example 8. Let V be the set of all positive real numbers. For $\vec{u} = u$ and $\vec{v} = v$ in V, we define addition and scalar multiplication by

$$\vec{u} + \vec{v} = uv \qquad \text{and} \qquad k\vec{u} = u^k.$$

Then V with these two operations is a vector space.

Some properties of vector spaces. Let V be a vector space, let $\vec{u}$ be a vector in V, and let k be a scaler. Then:

1. $0\vec{u} = \vec{0}$.
2. $k\vec{0} = \vec{0}$
3. $(-1)\vec{u} = -\vec{u}$
4. If $k\vec{u} = \vec{0}$, then either $k = 0$ or $\vec{u} = \vec{0}$.

Proof of 1.

Proof of 3.