Section 3.5: Cross Product

Objectives.

- Introduce the cross product of two vectors in $\mathbb{R}^3$.
- Interpret the cross product geometrically.
- Study some properties of the cross product.

The cross product of two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is

note: vector x vector = vector.

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

(Note that the cross product is only defined for vectors in $\mathbb{R}^3$.)

Example 1. Compute $\vec{u} \times \vec{v}$ for the vectors $\vec{u} = (2, 3, -2)$ and $\vec{v} = (1, 4, 1)$.

$$\vec{u} \times \vec{v} = \big((3)(1) - (-2)(4),\ (-2)(1) - (2)(1),\ (2)(4) - (3)(1)\big)$$

$$= (11, -4, 5).$$

The cross product can also be expressed as a 3×3 determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

Example 2. Compute $\vec{v} \times \vec{u}$ for the vectors in Example 1. What do you notice?

$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

$$\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 1 \\ 2 & 3 & -2 \end{vmatrix} = -8\vec{i} + 2\vec{j} + 3\vec{k} - 8\vec{k} - 3\vec{i} + 2\vec{j}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 4 & 1 & 1 & 4 \\ 2 & 3 & -2 & 2 & 3 \end{vmatrix}$$

$$= -11\vec{i} + 4\vec{j} - 5\vec{k}$$

$$= (-11, 4, -5).$$

Properties of the cross product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$ and k is a scalar, then:

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ anticommutative
2. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
3. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$

   cross products distributes over addition
4. $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$ scalar multiples behave "nicely"
5. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
6. $\vec{u} \times \vec{u} = \vec{0}$

Proof of 1. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = -(\vec{v} \times \vec{u}).$$

Example 3. Show that $(\vec{u} + k\vec{v}) \times \vec{v} = \vec{u} \times \vec{v}$.

$$(\vec{u} + k\vec{v}) \times \vec{v} = (\vec{u} \times \vec{v}) + (k\vec{v} \times \vec{v}) = (\vec{u} \times \vec{v}) + k(\cancelto{\vec{0}}{\vec{v} \times \vec{v}})$$

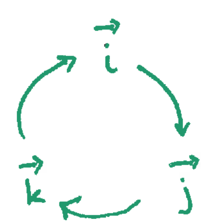
$$= (\vec{u} \times \vec{v}) + k\vec{0} = \vec{u} \times \vec{v}.$$

Example 4. Compute the following cross products, where $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$.

(a) $\vec{i} \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$

(b) $\vec{j} \times \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}$

(c) $\vec{k} \times \vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 0 & 0 \\ 1 & 0 \end{matrix} = \vec{j}$



$\vec{i} \times \vec{i} = \vec{0}$

$\vec{i} \times \vec{j} = \vec{k}$

$\vec{i} \times \vec{k} = -\vec{j}$

An important property of the cross product is that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Relationships between the dot product and the cross product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$, then:

1. $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ i.e. $\vec{u}$ is orthogonal to $\vec{u} \times \vec{v}$
2. $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \, \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$ Lagrange's identity
3. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ scalar triple product
4. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$ vector triple product

Proof of 1. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Then:

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = (u_1, u_2, u_3) \cdot (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$$

$$= u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_2 u_1 v_3 + u_3 u_1 v_2 - u_3 u_2 v_1$$

$$= 0.$$

Therefore $\vec{u}$ and $\vec{u} \times \vec{v}$ are orthogonal.

Example 5. For the vectors $\vec{u} = (2, 3, -2)$ and $\vec{v} = (1, 4, 1)$ in Example 1, confirm that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

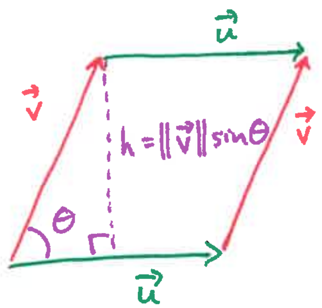
recall: $\vec{u} \times \vec{v} = (11, -4, 5)$.

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = (2, 3, -2) \cdot (11, -4, 5) = 22 - 12 - 10 = 0$$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = (1, 4, 1) \cdot (11, -4, 5) = 11 - 16 + 5 = 0$$

Thus both $\vec{u}$ and $\vec{v}$ are orthogonal to $\vec{u} \times \vec{v}$.

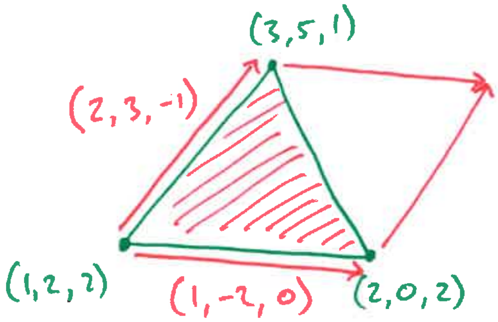
The norm of $\vec{u} \times \vec{v}$ is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.



(from Lagrange:)
$$\begin{aligned}\|\vec u \times \vec v\|^2 &= \|\vec u\|^2 \|\vec v\|^2 - (\vec u \cdot \vec v)^2 \\ &= \|\vec u\|^2 \|\vec v\|^2 - \|\vec u\|^2 \|\vec v\|^2 \cos^2\theta \\ &= \|\vec u\|^2 \|\vec v\|^2 (1 - \cos^2\theta) \\ &= \|\vec u\|^2 \|\vec v\|^2 \sin^2\theta\end{aligned}$$

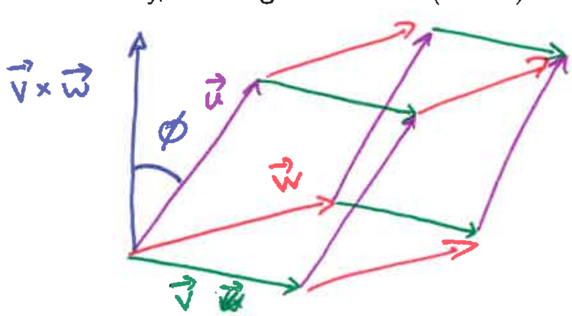
Therefore: $\boxed{\|\vec u \times \vec v\| = \|\vec u\| \|\vec v\| \sin\theta}$ ← area of parallelogram.

Example 6. Find the area of the triangle with vertices $(1, 2, 2)$, $(3, 5, 1)$, and $(2, 0, 2)$.



$$\begin{aligned}\text{area} &= \tfrac{1}{2} \|(1,-2,0) \times (2,3,-1)\| \\ &= \tfrac{1}{2} \|(2,1,7)\| \\ &= \tfrac{1}{2}\sqrt{54}.\end{aligned}$$

Similarly, the magnitude of $\vec{u} \cdot (\vec{v} \times \vec{w})$ is the volume of the parallelipiped spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$.



$$\begin{aligned}\text{volume} &= \text{height} \times \text{area of base} \\ &= \left(\|\vec u\| |\cos\phi|\right)\left(\|\vec v \times \vec w\|\right) \\ &= \|\vec u\| \|\vec v \times \vec w\| |\cos\phi| \\ &= |\vec u \cdot (\vec v \times \vec w)|\end{aligned}$$

Example 7. Find the volume of the parallelipiped spanned by $(1, 2, 2)$, $(3, 5, 1)$, and $(2, 0, 2)$.

$$\text{volume} = |(1,2,2) \cdot ((3,5,1) \times (2,0,2))| = |(1,2,2) \cdot (10,-4,-10)|$$
$$= |10 - 8 - 20| = |-18| = 18.$$

Theorem. The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ lie in the same plane if and only if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

i.e. the volume spanned by $\vec u, \vec v, \vec w$ is zero, so these vectors determine a flat surface rather than a parallelepiped.