Section 3.4: The Geometry of Linear Systems

Objectives.

- Write vector and parametric equations for lines and planes in $\mathbb{R}^n$.
- Express a line segment in vector form.

In Section 3.3, we saw how the dot product allows us to write vector and scalar equations for a line in $\mathbb{R}^2$ or a plane in $\mathbb{R}^3$. Specifically:

- the line in $\mathbb{R}^2$ through the point $\vec{x}_0 = (x_0, y_0)$ and normal to the vector $\vec{n} = (a, b)$ is

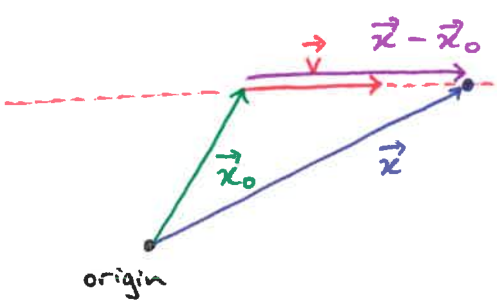
$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \quad \underline{\text{or}} \quad a(x - x_0) + b(y - y_0) = 0.$$

- the plane in $\mathbb{R}^3$ through the point $\vec{x}_0 = (x_0, y_0, z_0)$ and normal to the vector $\vec{n} = (a, b, c)$ is

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \quad \underline{\text{or}} \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

In this section, we will explore how the equation of a line in higher dimensions can be written using a point on the line and a direction parallel to the line, and how the equation of a plane in higher dimensions can be written using a point on the plane and two (non-parallel!) directions parallel to the plane.

Suppose that $\vec{x}$ is a general point on the line through the point $\vec{x}_0$ and parallel to the vector $\vec{v}$.



A vector on this line is a scalar multiple of $\vec{v}$.

$$\vec{x} - \vec{x}_0 = t\,\vec{v}$$

parameter

$$\Rightarrow \quad \vec{x} = \vec{x}_0 + t\vec{v}$$

gen. pt. = fixed pt. + parameter $\cdot$ direction

Example 1. Let L be the line in $\mathbb{R}^3$ through the point $\vec{x}_0 = (3, -1, 5)$ and parallel to the vector $\vec{v} = (-2, 1, 2)$.

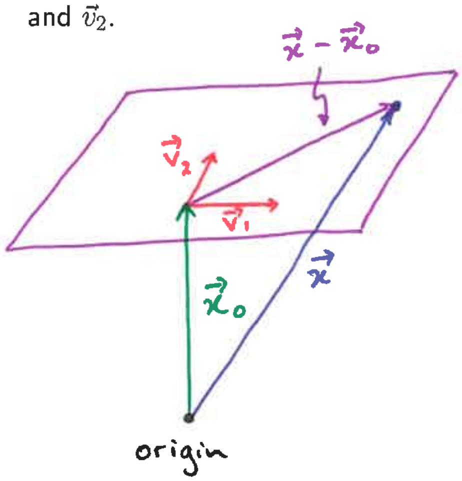
(a) Find a vector equation for the line L.

$$\vec{x} = \vec{x}_0 + t\vec{v} = (3, -1, 5) + t(-2, 1, 2). = (3 - 2t, -1 + t, 5 + 2t).$$

(b) Find parametric equations for the line L.

$$x = 3 - 2t, \quad y = -1 + t, \quad z = 5 + 2t$$

Suppose $\vec{x}$ is a general point on the plane through the point $\vec{x}_0$ and parallel to the (non-parallel) vectors $\vec{v}_1$ and $\vec{v}_2$.



A vector $\vec{x}-\vec{x}_0$ in the plane is a linear combination of $\vec{v}_1$ and $\vec{v}_2$

$$\vec{x}-\vec{x}_0 = t_1\vec{v}_1 + t_2\vec{v}_2$$

$$\Rightarrow \vec{x} = \vec{x}_0 + t_1\vec{v}_1 + t_2\vec{v}_2.$$

Example 2. Consider the point $\vec{x}_0 = (1,4,0,-3)$ in $\mathbb{R}^4$ and the vectors $\vec{v}_1 = (2,-1,1,0)$ and $\vec{v}_2 = (3,-6,5,2)$.

(a) Find a vector equation for the plane through $\vec{x}_0$ and parallel to both $\vec{v}_1$ and $\vec{v}_2$.

$$\vec{x} = \vec{x}_0 + t_1\vec{v}_1 + t_2\vec{v}_2 = (1,4,0,-3) + t_1(2,-1,1,0) + t_2(3,-6,5,2).$$

(b) Find parametric equations for the plane in part (a).

$$w = 1 + 2t_1 + 3t_2, \quad x = 4 - t_1 - 6t_2, \quad y = t_1 + 5t_2, \quad z = -3 + 2t_2.$$

Example 3. The scalar equation $x + 2y + 3z = 4$ represents a plane in $\mathbb{R}^3$.

(a) Find parametric equations for the plane. $\rightarrow$ use two variables as parameters!!!

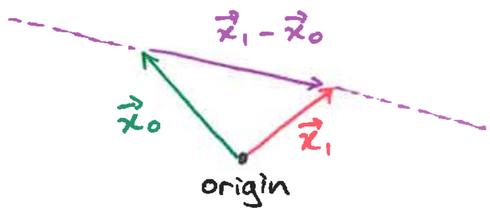
Let $y = t_1$ and $z = t_2$. Then

$$x = 4 - 2t_1 - 3t_2.$$

(b) Find a vector equation for the plane.

$$\vec{x} = (4-2t_1-3t_2,\ t_1,\ t_2) = (4,0,0) + t_1(-2,1,0) + t_2(-3,0,1).$$

Any two distinct points $\vec{x}_0$ and $\vec{x}_1$ in $\mathbb{R}^n$ determine a unique line:



$$\vec{x} = \vec{x}_0 + t\underline{(\vec{x}_1 - \vec{x}_0)}$$

i.e. $\vec{v} = \vec{x}_1 - \vec{x}_0$ is a direction parallel to the line.

or

$$\vec{x} = (1-t)\vec{x}_0 + t\vec{x}_1$$

Example 4. Consider the two points $\vec{x}_0 = (1, -1)$ and $\vec{x}_1 = (0, 3)$ in $\mathbb{R}^2$.

(a) Find a vector equation for the line through $\vec{x}_0$ and $\vec{x}_1$.

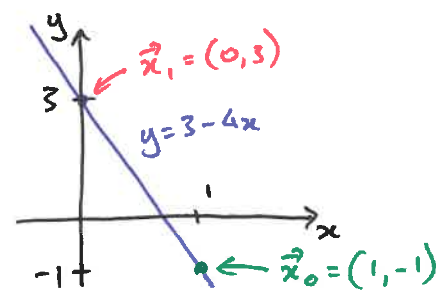
$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) = (1,-1) + t(0-1,\ 3-(-1)) = (1,-1) + t(-1,4).$$

(b) Write a scalar equation for the line in part (a).

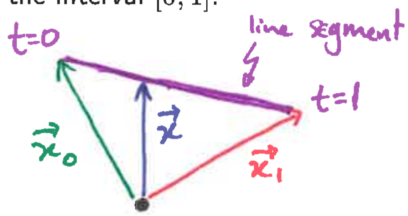
From $x = 1-t$ and $y = -1+4t$, we have

$t = 1-x$ and $4t = 1+y$.

Thus $4(1-x) = 1+y$, or $\underline{y = 3-4x}$.



To describe the line segment connecting two points $\vec{x}_0$ and $\vec{x}_1$ in $\mathbb{R}^n$, we can restrict the values of the parameter t to the interval $[0, 1]$:



$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0), \quad 0 \le t \le 1$$

or

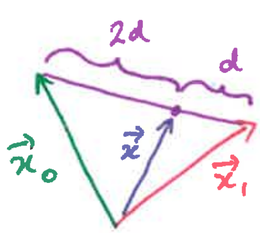
$$\vec{x} = (1-t)\vec{x}_0 + t\vec{x}_1, \quad 0 \le t \le 1$$

Example 5. Consider the two points $\vec{x}_0 = (1, -4, -2, 5)$ and $\vec{x}_1 = (4, -2, 7, 2)$.

(a) Find an equation for the line segment from $\vec{x}_0$ to $\vec{x}_1$.

$$\vec{x} = (1-t)(1,-4,-2,5) + t(4,-2,7,2), \quad 0 \le t \le 1.$$

(b) Find the point on this line segment for which the distance to $\vec{x}_0$ is twice the distance to $\vec{x}_1$.



- use $t = \frac{2}{3}$ (i.e. $\frac{2}{3}$ of distance from $\vec{x}_0$ to $\vec{x}_1$)

$$\vec{x} = \left(1-\tfrac{2}{3}\right)(1,-4,-2,5) + \tfrac{2}{3}(4,-2,7,2)$$

$$= \left(\tfrac{1}{3}, -\tfrac{4}{3}, -\tfrac{2}{3}, \tfrac{5}{3}\right) + \left(\tfrac{8}{3}, -\tfrac{4}{3}, \tfrac{14}{3}, \tfrac{4}{3}\right)$$

$$= \left(3, -\tfrac{8}{3}, 4, 3\right).$$

Recall that a homogeneous linear equation has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

or: $\vec{a} \cdot \vec{x} = 0$, where $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$.

Notice from this that every vector that satisfies a homogeneous linear equation is orthogonal to the coefficient vector. In particular, any solution to the matrix equation $A\vec{x} = \vec{0}$ is orthogonal to every row of the matrix A.

Theorem. If A is an $m \times n$ matrix, then the set of solutions to the homogeneous linear system $A\vec{x} = \vec{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row of A.

Example 6. The linear system

$$\begin{bmatrix} 1 & 5 & -10 & 0 & 2 \\ 3 & -2 & 0 & 2 & 1 \\ 4 & 2 & 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solution $x_1 = -2t$, $x_2 = 2s$, $x_3 = s + t$, $x_4 = 2s$, $x_5 = 6t$. Show that the vector

$$\vec{x} = (-2t, 2s, s+t, 2s, 6t) \leftarrow \text{solutions for system!!!}$$

is orthogonal to every row of the coefficient matrix for the system.

$$\vec{r}_1 \cdot \vec{x} = (1, 5, -10, 0, 2) \cdot (-2t, 2s, s+t, 2s, 6t)$$
$$= -2t + 10s - 10(s+t) + 0(2s) + 2(6t)$$
$$= -2t + 10s - 10s - 10t + 12t = \underline{0.}$$

$$\vec{r}_2 \cdot \vec{x} = (3, -2, 0, 2, 1) \cdot (-2t, 2s, s+t, 2s, 6t)$$
$$= 3(-2t) - 2(2s) + 0(s+t) + 2(2s) + 1(6t)$$
$$= -6t - 4s + 4s + 6t = \underline{0.}$$

$$\vec{r}_3 \cdot \vec{x} = (4, 2, 2, -3, 1) \cdot (-2t, 2s, s+t, 2s, 6t)$$
$$= -8t + 4s + 2s + 2t - 6s + 6t = \underline{0.}$$