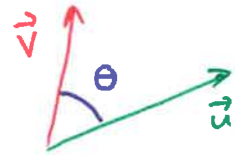
Section 3.3: Orthogonality

Objectives.

- Introduce the definition of orthogonality in $\mathbb{R}^n$.
- Represent lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$ using vector equations.
- Project a vector onto a line.
- Write a vector as the sum of two orthogonal components.

In Section 3.2, we defined the angle θ between two vectors $\vec{u}$ and $\vec{v}$ as

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$



The vectors $\vec{u}$ and $\vec{v}$ are orthogonal (or perpendicular) if

$$\vec{u}\cdot\vec{v} = 0.$$

note:
$\vec{u}\cdot\vec{v} > 0 \implies \theta$ is acute
$\vec{u}\cdot\vec{v} = 0 \implies \theta$ is a right angle
$\vec{u}\cdot\vec{v} < 0 \implies \theta$ is obtuse

Example 1. Show that the vectors $\vec{u} = (1, -2, 2, 5)$ and $\vec{v} = (3, 2, 3, -1)$ in $\mathbb{R}^4$ are orthogonal.

$$\vec{u}\cdot\vec{v} = (1,-2,2,5)\cdot(3,2,3,-1) = 3-4+6-5 = 0.$$

Thus $\vec{u}$ and $\vec{v}$ are orthogonal.

Notice that in $\mathbb{R}^n$, the standard basis vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are all orthogonal.

eg. $\vec{e}_1 \cdot \vec{e}_n = (1,0,\cdots,0)\cdot(0,0,\ldots,1) = 0.$

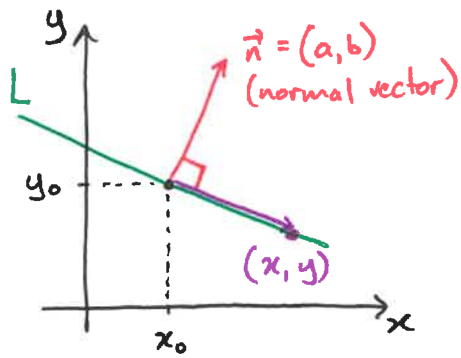
Pythagorean Theorem in $\mathbb{R}^n$. If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ then

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Proof. $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = (\vec{u}\cdot\vec{u}) + 2\underbrace{(\vec{u}\cdot\vec{v})}_{=0} + (\vec{v}\cdot\vec{v})$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2.$

A straight line in $\mathbb{R}^2$ can be described by specifying a point and a normal direction (that is, a vector orthogonal to the line).



If (x,y) is any point on the line L, then $(x-x_0, y-y_0)$ is orthogonal to $\vec{n}$.

$$\vec{n}\cdot(x-x_0, y-y_0) = 0$$

$$(a,b)\cdot(x-x_0, y-y_0) = 0$$

$$\boxed{a(x-x_0) + b(y-y_0) = 0}$$

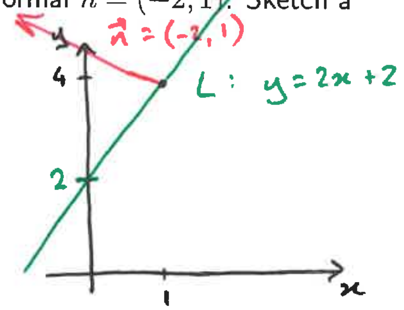
or: $ax + by + c = 0.$

Example 2. Write an equation for the line in $\mathbb{R}^2$ through the point $(1,4)$ with normal $\vec{n} = (-2,1)$. Sketch a diagram indicating the point, the normal vector, and the line.

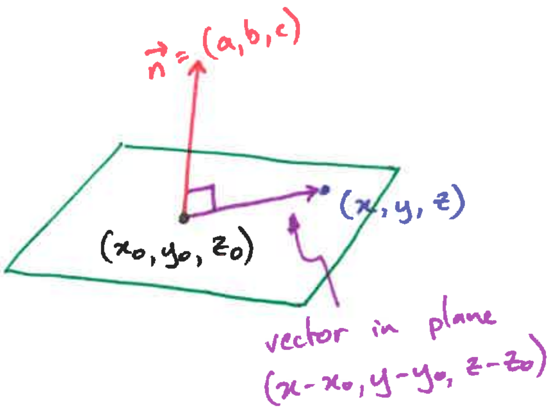
$$\vec{n}\cdot(x-x_0, y-y_0) = 0 \Rightarrow (-2,1)\cdot(x-1, y-4) = 0$$

$$\Rightarrow -2(x-1) + 1(y-4) = 0$$

$$\Rightarrow \underline{-2x + y - 2 = 0.}$$



The same idea can be used to write equations for planes in $\mathbb{R}^3$.



$$\vec{n}\cdot(x-x_0, y-y_0, z-z_0) = 0$$

$$(a,b,c)\cdot(x-x_0, y-y_0, z-z_0) = 0$$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

or: $ax + by + cz + d = 0$

Example 3. Write an equation for the plane in $\mathbb{R}^3$ through the point $(2,-5,0)$ with normal $\vec{n} = (1,3,-1)$.

$$\vec{n}\cdot(x-x_0, y-y_0, z-z_0) = 0 \Rightarrow (1,3,-1)\cdot(x-2, y+5, z) = 0$$

$$\Rightarrow (x-2) + 3(y+5) - z = 0$$

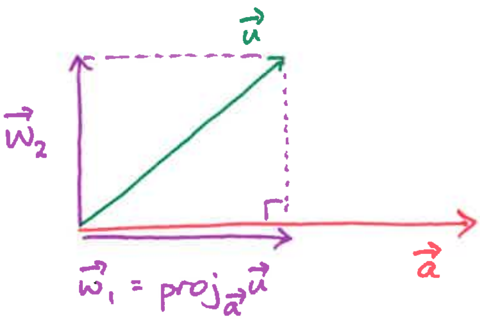
$$\Rightarrow \underline{x + 3y - z + 13 = 0.}$$

In Chapter 1, we introduced (orthogonal) projections onto the coordinate axes as examples or linear transformations. We can now extend this idea to (orthogonal) projections onto any line in $\mathbb{R}^n$.

Projection Theorem. If $\vec{u}$ and $\vec{a}$ are vectors in $\mathbb{R}^n$ with $\vec{a} \neq \vec{0}$, then $\vec{u}$ can be written in exactly one way as $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\vec{w}_1$ is parallel to $\vec{a}$ and $\vec{w}_2$ is orthogonal to $\vec{a}$. Specifically:

$$\vec{w}_1 = \text{proj}_{\vec{a}}\vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\,\vec{a} \qquad \text{and} \qquad \vec{w}_2 = \vec{u} - \text{proj}_{\vec{a}}\vec{u} = \vec{u} - \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\,\vec{a}.$$

why? $\vec{w}_1 = k\vec{a}$ and $\vec{w}_2 \cdot \vec{a} = 0$, so



$$\vec{u} \cdot \vec{a} = (\vec{w}_1 + \vec{w}_2) \cdot \vec{a} = \vec{w}_1 \cdot \vec{a} + \vec{w}_2 \cdot \vec{a}$$
$$= k\vec{a} \cdot \vec{a} + 0 = k\|\vec{a}\|^2.$$
$$\Longrightarrow \quad k = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}, \text{ so}$$
$$\vec{w}_1 = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}, \quad \vec{w}_2 = \vec{u} - \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}.$$

Example 4. Let $\vec{u} = (1, 2, 3)$ and $\vec{a} = (4, -1, -1)$. Find the component of $\vec{u}$ parallel to $\vec{a}$ and the component of $\vec{u}$ orthogonal to $\vec{a}$.

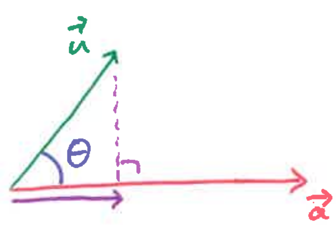
component $\parallel$ to $\vec{a}$:

$$\text{proj}_{\vec{a}}\vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = \frac{(1,2,3)\cdot(4,-1,-1)}{(4,-1,-1)\cdot(4,-1,-1)}(4,-1,-1) = \frac{-1}{18}(4,-1,-1) = \left(-\frac{2}{9}, \frac{1}{18}, \frac{1}{18}\right).$$

component $\perp$ to $\vec{a}$:

$$\vec{u} - \text{proj}_{\vec{a}}\vec{u} = (1,2,3) - \left(-\frac{2}{9}, \frac{1}{18}, \frac{1}{18}\right) = \left(\frac{11}{9}, \frac{35}{18}, \frac{53}{18}\right).$$

The norm of the orthogonal projection (of $\vec{u}$ onto $\vec{a}$ can be written either in terms of the two vectors or in terms of $\vec{u}$ and the angle θ between $\vec{u}$ and $\vec{a}$.



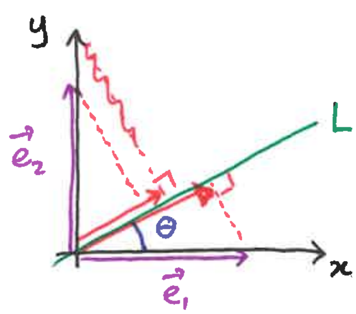
$\|\text{proj}_{\vec{a}}\vec{u}\| = \|\vec{u}\| \cos\theta$

$$\|\text{proj}_{\vec{a}}\vec{u}\| = \left\| \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} \right\| = \frac{|\vec{u} \cdot \vec{a}|}{\|\vec{a}\|^2}\|\vec{a}\| = \frac{|\vec{u} \cdot \vec{a}|}{\|\vec{a}\|}$$
$$= \frac{\|\vec{u}\|\|\vec{a}\| \cos\theta}{\|\vec{a}\|} = \|\vec{u}\| \cos\theta.$$

assuming θ is acute!!!

Example 5. Let L be a line through the origin in $\mathbb{R}^2$ that makes an angle θ with the positive x-axis.

(a) Find the projections of $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$ onto L.



$\vec{a} = (\cos\theta, \sin\theta)$ is a vector in the direction of L.

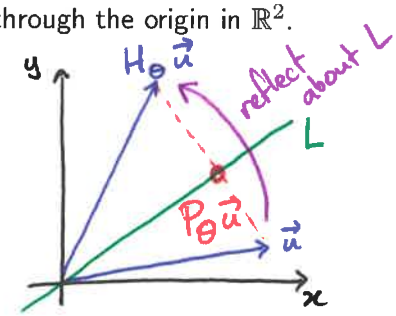
$$\text{proj}_{\vec{a}}\, \vec{e}_1 = \frac{(1,0)\cdot(\cos\theta, \sin\theta)}{1^2}(\cos\theta, \sin\theta) = (\cos^2\theta, \cos\theta\sin\theta)$$

$$\text{proj}_{\vec{a}}\, \vec{e}_2 = \frac{(0,1)\cdot(\cos\theta, \sin\theta)}{1^2}(\cos\theta, \sin\theta) = (\cos\theta\sin\theta, \sin^2\theta).$$

(b) Find the standard matrix P_θ for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that projects each point onto L.

$$P_\theta = \left[\ \text{proj}_{\vec{a}}\, \vec{e}_1 \ \middle|\ \text{proj}_{\vec{a}}\, \vec{e}_2\ \right] = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}.$$

We can use the previous example to find a linear transformation that reflects a vector/point about a line through the origin in $\mathbb{R}^2$.



$$P_\theta \vec{u} = \frac{1}{2}(H_\theta \vec{u} + \vec{u})$$

$$\Longrightarrow \quad P_\theta = \frac{1}{2}(H_\theta + I)$$

$$\Longrightarrow \quad H_\theta = 2P_\theta - I = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Example 6. Let $\vec{x} = (4,1)$ and let L be the line through the origin that makes an angle of $\pi/3$ with the positive x-axis.

(a) Find the projection of $\vec{x}$ onto L.

$$P_{\frac{\pi}{3}} = \begin{bmatrix} \cos^2\frac{\pi}{3} & \cos\frac{\pi}{3}\sin\frac{\pi}{3} \\ \cos\frac{\pi}{3}\sin\frac{\pi}{3} & \sin^2\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}, \quad \text{so} \quad P_{\frac{\pi}{3}}(4,1) = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\sqrt{3}}{4} \\ \sqrt{3} + \frac{3}{4} \end{bmatrix}$$

(b) Find the reflection of $\vec{x}$ about L.

$$H_{\frac{\pi}{3}} = \begin{bmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \text{so} \quad H_{\frac{\pi}{3}}(4,1) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + \frac{\sqrt{3}}{2} \\ 2\sqrt{3} + \frac{1}{2} \end{bmatrix}$$

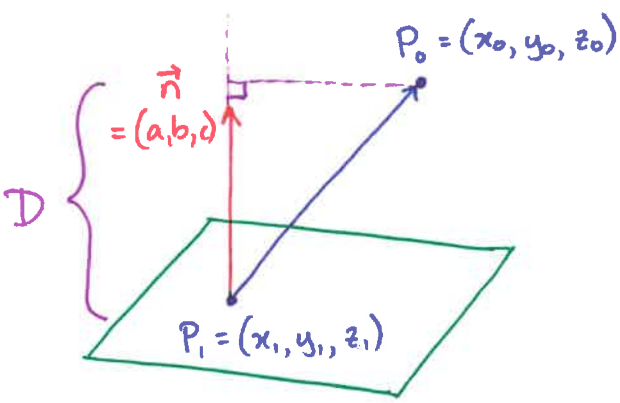
Distance problems.

The distance between a point and a line in $\mathbb{R}^2$ or between a point and a plane in $\mathbb{R}^3$ can be found using projections.

> **Theorem.**
>
> 1. In $\mathbb{R}^2$, the distance between the point $P_0 = (x_0, y_0)$ and the line $ax + by + c = 0$ is
> $$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$
>
> normal is $\vec{n} = (a,b,c)$
>
> 2. In $\mathbb{R}^3$, the distance between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is
> $$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Proof of 2.

Choose $P_1 = (x_1, y_1, z_1)$ in the plane, and project $\overrightarrow{P_1P_0}$ onto $\vec{n}$.



$$D = \| \text{proj}_{\vec{n}} \overrightarrow{P_1P_0} \|$$
$$= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a,b,c)|}{\|(a,b,c)\|}$$
$$= \frac{|ax_0 - ax_1 + by_0 - by_1 + cz_0 - cz_1|}{\sqrt{a^2+b^2+c^2}}$$
$$= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}.$$

b/c P_1 is in the plane,

$ax_1 + by_1 + cz_1 + d = 0.$

$\Rightarrow -ax_1 - by_1 - cz_1 = d.$

Example 7. Find the distance in $\mathbb{R}^2$ between the point $(1, -1)$ and the line $x + 2y = 3$.

$a = 1,\ b = 2,\ c = -3$

$$D = \frac{|1(1) + 2(-1) + (-3)|}{\sqrt{1^2 + 2^2}} = \frac{|-4|}{\sqrt{5}} = \frac{4}{\sqrt{5}}.$$