Section 3.2: Norm, Dot Product, and Distance in $\mathbb{R}^n$

Objectives.

- Define and apply the notions of norm and distance in $\mathbb{R}^n$.
- Introduce the dot product of two vectors, and interpret the dot product geometrically.
- Study some properties and applications of the dot product.

The norm (length, magnitude) of a vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

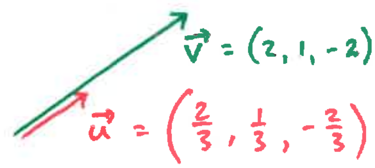
note: this generalizes Pythagoras!!!

Dividing a (non-zero) vector $\vec{v}$ by its norm produces the unit vector in the same direction as $\vec{v}$.

Example 1. Find the unit vector $\vec{u}$ that has the same direction as $\vec{v} = (2, 1, -2)$. Check that $\|\vec{u}\| = 1$.

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3.$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{3}(2, 1, -2) = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right).$$



check: $\|\vec{u}\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{9}{9}} = 1.$

The distance between two points $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Example 2. Find the distance between the points $\vec{u} = (1, 3, -2, 0, 2)$ and $\vec{v} = (3, 0, 1, 1, -1)$ in $\mathbb{R}^5$.

$$d(\vec{u}, \vec{v}) = \sqrt{(1-3)^2 + (3-0)^2 + (-2-1)^2 + (0-1)^2 + (2-(-1))^2}$$

$$= \sqrt{4 + 9 + 9 + 1 + 9}$$

$$= \sqrt{32}$$

$$= 4\sqrt{2}.$$

The dot product of two vectors $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

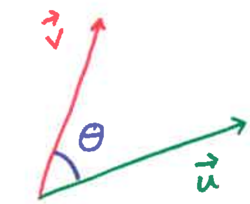
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

note: vector · vector = scalar.

Example 3. Find the dot product of the vectors $\vec{u} = (1, 3, 2, 4)$ and $\vec{v} = (-1, 1, -2, 1)$

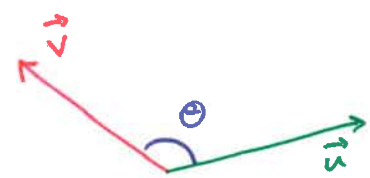
$$\vec{u} \cdot \vec{v} = (1,3,2,4) \cdot (-1,1,-2,1)$$
$$= -1 + 3 - 4 + 4$$
$$= \underline{2}.$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the dot product of two vectors is related to the angle between them. (This can also be generalized to finding "angles" between vectors in higher-dimensional spaces.)



$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

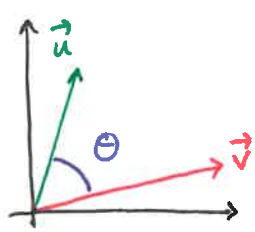
$$\Rightarrow \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$



θ acute $\Leftrightarrow$ $\vec{u} \cdot \vec{v} > 0$.

θ obtuse $\Leftrightarrow$ $\vec{u} \cdot \vec{v} < 0$.

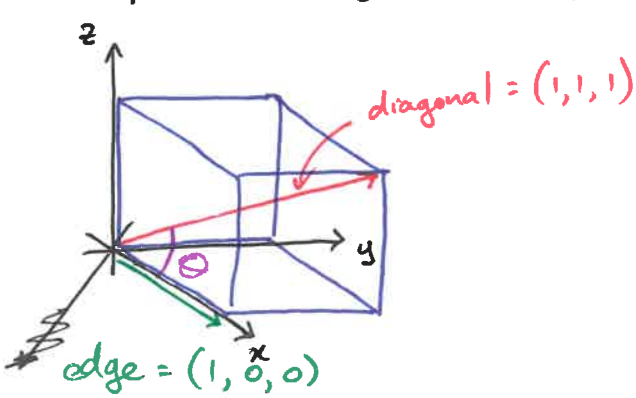
Example 4. Find the angle between the vectors $\vec{u} = (1, 2)$ and $\vec{v} = (3, 1)$.



$\vec{u} \cdot \vec{v} = 5$, $\quad \|\vec{u}\| = \sqrt{5}$, $\quad \|\vec{v}\| = \sqrt{10}$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}} \Rightarrow \underline{\theta = 45^\circ}.$$

Example 5. Find the angle between a diagonal and an edge of a cube.



$$\cos\theta = \frac{(1,1,1) \cdot (1,0,0)}{\|(1,1,1)\| \, \|(1,0,0)\|}$$
$$= \frac{1}{\sqrt{3}}$$
$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \underline{54.74^\circ}.$$

Notice that the dot product of a vector with itself is the square of the norm of the vector.

If $\vec{v} = (v_1, v_2, \cdots, v_n)$ is a vector in $\mathbb{R}^n$, then

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\vec{v}\|^2.$$

Properties of the dot product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and k is a scalar, then:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ "symmetry" (dot product commutes)
2. $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} = 0$ ← zero scalar (zero vector)
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
4. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ } dot product distributes over addition
5. $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v})$ "homogeneity"
6. $\vec{v} \cdot \vec{v} \geq 0$, and $\vec{v} \cdot \vec{v} = 0$ if and only if $\vec{v} = \vec{0}$. "positivity"

Example 6. Use properties 1 and 3 above to prove property 4.

$$\begin{aligned}(\vec{u} + \vec{v}) \cdot \vec{w} &= \vec{w} \cdot (\vec{u} + \vec{v}) && \text{by property 1}\\ &= \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v} && \text{by property 3}\\ &= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} && \text{by property 1.}\end{aligned}$$

Example 7. Expand and simplify the vector expression.

$$\begin{aligned}(2\vec{u} + 3\vec{v}) \cdot (3\vec{u} - \vec{v}) &= 2\vec{u} \cdot (3\vec{u} - \vec{v}) + 3\vec{v} \cdot (3\vec{u} - \vec{v})\\ &= 6(\vec{u} \cdot \vec{u}) - 2(\vec{u} \cdot \vec{v}) + 9(\vec{v} \cdot \vec{u}) - 3(\vec{v} \cdot \vec{v})\\ &= 6\|\vec{u}\|^2 - 2(\vec{u} \cdot \vec{v}) + 9(\vec{u} \cdot \vec{v}) - 3\|\vec{v}\|^2\\ &= 6\|\vec{u}\|^2 + 7(\vec{u} \cdot \vec{v}) - 3\|\vec{v}\|^2.\end{aligned}$$

There are two important inequalities involving norms and distances in $\mathbb{R}^n$.

Cauchy-Schwarz Inequality. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

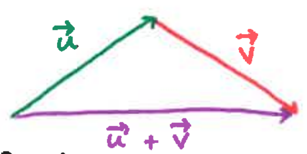
$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|.$$

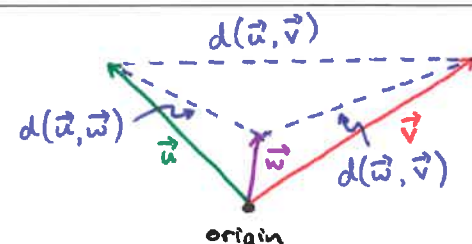
note: this implies that $-1 \leq \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \leq 1$, so we can define the angle between $\vec{u}$ and $\vec{v}$ as $\theta = \cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$.

Triangle Inequality. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, then:

(a) $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ triangle inequality for vectors

(b) $d(\vec{u},\vec{v}) \leq d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$ triangle inequality for distances





Proof of (a).

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \quad \leftarrow \text{because } \|\vec{a}\|^2 = \vec{a}\cdot\vec{a} \\
&= (\vec{u}\cdot\vec{u}) + 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v}) \\
&\leq \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2 \quad \text{apply absolute value to } \vec{u}\cdot\vec{v} \\
&\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2.
\end{aligned}$$

Because $\|\vec{u}+\vec{v}\| \geq 0$ and $\|\vec{u}\|+\|\vec{v}\| \geq 0$, we have $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\|+\|\vec{v}\|$.

Example 8. Suppose that $\|\vec{u}\| = 4$ and $\|\vec{v}\| = 3$. What are the smallest and largest possible values of $\|\vec{u}+\vec{v}\|$?

$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| = 4+3 = 7.$

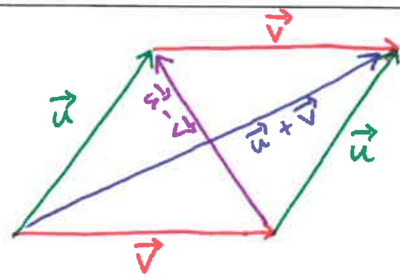
$\|\vec{u}\| = \|(\vec{u}+\vec{v})-\vec{v}\| \leq \|\vec{u}+\vec{v}\| + \|\vec{v}\|$, so $4 \leq \|\vec{u}+\vec{v}\| + 3$.

Thus $\|\vec{u}+\vec{v}\| \geq 1$, and therefore $1 \leq \|\vec{u}+\vec{v}\| \leq 7$.

In plane geometry (that is, in $\mathbb{R}^2$), the sum of the squares of the two diagonals of a parallelogram equals the sum of the squares of the four sides. This result is also true more generally in $\mathbb{R}^n$.

Parallelogram equation for vectors. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).$$



Proof.

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\
&= (\vec{u}\cdot\vec{u}) + 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v}) + (\vec{u}\cdot\vec{u}) - 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v}) \\
&= 2(\vec{u}\cdot\vec{u}) + 2(\vec{v}\cdot\vec{v}) \\
&= 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).
\end{aligned}$$

Taking the difference of the squares of the two diagonals of a parallelogram instead gives a different expression for the dot product of two vectors.

Theorem. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2.$$

Proof.

$$\begin{aligned}
\tfrac{1}{4}\|\vec{u}+\vec{v}\|^2 - \tfrac{1}{4}\|\vec{u}-\vec{v}\|^2 &= \tfrac{1}{4}(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) - \tfrac{1}{4}(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\
&= \tfrac{1}{4}\left((\vec{u}\cdot\vec{u}) + 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v})\right) - \tfrac{1}{4}\left((\vec{u}\cdot\vec{u}) - 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v})\right) \\
&= \tfrac{1}{4}\left(4(\vec{u}\cdot\vec{v})\right) \\
&= \vec{u}\cdot\vec{v}.
\end{aligned}$$