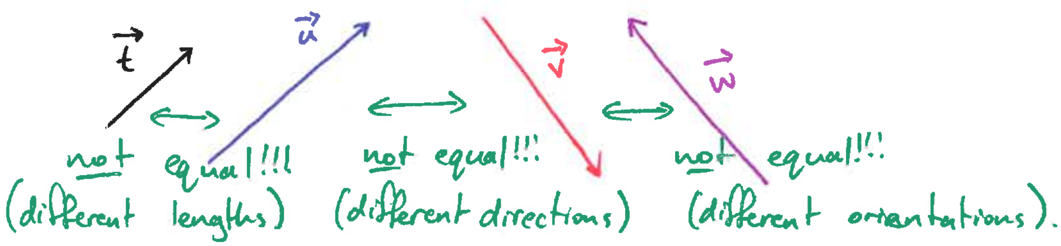
Section 3.1: Vectors in 2-space, 3-space, and n-space

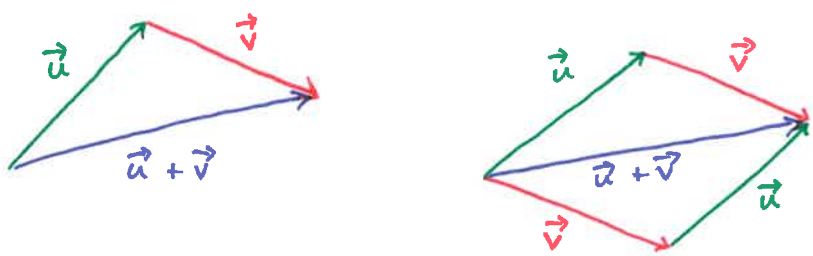
Objectives.

- Introduce the some terminology and notation for vectors.
- Understand vector operations in $\mathbb{R}^n$ geometrically and algebraically.
- Study some properties of vector operations.

A (geometric) vector is a quantity with a direction and a length, often represented by an arrow.

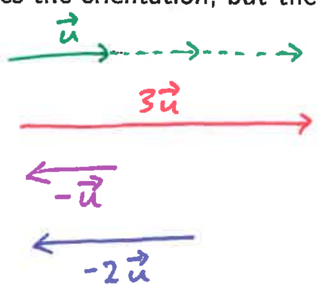


Two vectors can be added (geometrically) by placing the vectors end-to-end. (This is referred to as either the "triangle rule" or the "parallelogram rule".)



note: $\vec{u}+\vec{v}$ is the diagonal of a parallelogram with sides $\vec{u}$ and $\vec{v}$.

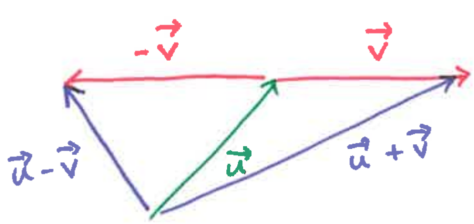
Multiplying a vector by a scalar changes ("scales") the length of the vector without changing the direction. If one vector is a scalar multiple of another, then we say the vectors are parallel. (Multiplying by a negative scalar reverses the orientation, but the result is still parallel to the original vector.)



$\vec{u}, 3\vec{u}, -\vec{u}, -2\vec{u}$ are all parallel.

note: the zero vector $\vec{0}$ is "parallel to" every vector!!!

We can view subtraction of a vector as "adding the negative of the vector".



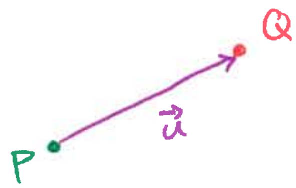
If $\underbrace{P = (a_1, a_2, \ldots, a_n)}_{\text{initial point}}$ and $\underbrace{Q = (b_1, b_2, \ldots, b_n)}_{\text{terminal point}}$ are two points in $\mathbb{R}^n$, then the vector from P to Q is

$$\overrightarrow{PQ} = (b_1 - a_1, b_2 - a_2, \cdots, b_n - a_n).$$

Two vectors $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ are equal if their components are equal. That is:

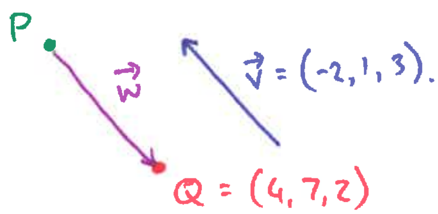
$$\vec{u} = \vec{v} \iff u_1 = v_1 \text{ and } u_2 = v_2 \text{ and } \cdots \text{ and } u_n = v_n.$$

Example 1. Find the vector $\vec{u} = \overrightarrow{PQ}$ that has initial point $P = (3, -1)$ and terminal point $Q = (-2, 8)$.



$$\vec{u} = \overrightarrow{PQ} = (-2-3, 8-(-1)) = \underline{(-5, 9)}.$$

Example 2. Find the initial point of a vector $\vec{w}$ that has terminal point $Q = (4, 7, 2)$ and is parallel to $\vec{v} = (-2, 1, 3)$ but has the opposite orientation. i.e. choose $\vec{w} = k\vec{v}$ where $k < 0$.



$$P = (4 + (-2), 7+1, 2+3) = \underline{(2, 8, 5)}.$$

Arithmetic with vectors (addition, subtraction, scalar multiplication) is done componentwise. If $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$ and k is a scalar, then we define:

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$
$$k\vec{u} = (ku_1, ku_2, \ldots, ku_n)$$
$$-\vec{u} = (-u_1, -u_2, \cdots, -u_n)$$

Example 3. Let $\vec{u} = (3, 1, 4, -2)$ and $\vec{v} = (1, -2, 3, 0)$. Simplify:

(a) $\vec{u} + \vec{v} = (3,1,4,-2) + (1,-2,3,0) = \underline{(4,-1,7,-2)}.$

(b) $3\vec{u} - 4\vec{v} = 3(3,1,4,-2) - 4(1,-2,3,0) = (9,3,12,-6) - (4,-8,12,0)$

$= \underline{(5, 11, 0, -6)}.$

Properties of vector operations. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and k and m are scalars, then:

1. $(\vec{u}+\vec{v})+\vec{w} = \vec{u} + (\vec{v} + \vec{w})$
2. $\vec{u}+\vec{v} = \vec{v} + \vec{u}$
3. $\vec{u}+\vec{0} = \vec{u}$
4. $\vec{u}+(-\vec{u}) = \vec{0}$
5. $k(\vec{u}+\vec{v}) = k\vec{u} + k\vec{v}$
6. $(k+m)\vec{u} = k\vec{u} + m\vec{u}$
7. $k(m\vec{u}) = (km)\vec{u} = m(k\vec{u})$
8. $1\vec{u} = \vec{u}$

Proof of 2. Let $\vec{u} = (u_1, u_2, \cdots, u_n)$ and $\vec{v} = (v_1, v_2, \cdots, v_n)$. Then

$$\begin{aligned}
\vec{u} + \vec{v} &= (u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) && \text{def. of vector addition} \\
&= (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n) && \text{addition in } \mathbb{R} \text{ is commutative.} \\
&= (v_1 + u_1, v_2 + u_2, \cdots, v_n + u_n) && \text{def. of vector addition.} \\
&= (v_1, v_2, \cdots, v_n) + (u_1, u_2, \cdots, u_n) \\
&= \vec{v} + \vec{u}.
\end{aligned}$$

Example 4. Let $\vec{u} = (-1, 4, 6)$ and $\vec{v} = (3, 3, 3)$. Find the vector $\vec{x}$ satisfying $4\vec{x} - 2\vec{u} = 2\vec{x} - \vec{v}$.

$$4\vec{x} - 2\vec{u} = 2\vec{x} - \vec{v} \implies 2\vec{x} = 2\vec{u} - \vec{v}$$

$$\implies \vec{x} = \tfrac{1}{2}(2\vec{u} - \vec{v}) = \vec{u} - \tfrac{1}{2}\vec{v}$$

$$= (-1, 4, 6) - \tfrac{1}{2}(3, 3, 3)$$

$$= \left(-\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{9}{2}\right).$$

zero scalar   zero vector

Theorem. If $\vec{v}$ is a vector in $\mathbb{R}^n$ and k is a scalar, then

1. $0\vec{v} = \vec{0}$
2. $k\vec{0} = \vec{0}$
3. $(-1)\vec{v} = -\vec{v}$

Proof of 1. Let $\vec{v} = (v_1, v_2, \cdots, v_n)$. Then:

$$0\vec{v} = 0(v_1, v_2, \cdots, v_n) = (0v_1, 0v_2, \cdots, 0v_n) = (0, 0, \cdots, 0) = \vec{0}.$$

A vector $\vec{w}$ in $\mathbb{R}^n$ is a <u>linear combination</u> of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r} \in \mathbb{R}^n$ if

$$\vec{w} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + \cdots + k_r \vec{v}_r, \quad \text{where } k_1, k_2, \ldots, k_r \text{ are scalars.}$$

Example 5. Find scalars c_1, c_2, c_3 satisfying $c_1(1,2,2) + c_2(0,1,-1) + c_3(3,1,2) = (-1,7,7)$.

- i.e. write $(-1,7,7)$ as a linear combination of $(1,2,2)$, $(0,1,-1)$, $(3,1,2)$.

This equation is equivalent to the linear system

$$\begin{aligned} c_1 \phantom{{}+c_2} + 3c_3 &= -1 \\ 2c_1 + c_2 + c_3 &= 7 \\ 2c_1 - c_2 + 2c_3 &= 7 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 2 & 1 & 1 & 7 \\ 2 & -1 & 2 & 7 \end{array}\right].$$

We can reduce this to rref using Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]. \quad \leftarrow \text{rref. for linear system.}$$

That is, $c_1 = 5$, $c_2 = -1$, $c_3 = -2$.

Example 6. Show that there is no choice of scalars a and b such that $a(3,-6) + b(-1,2) = (1,1)$.

We need to solve the system

$$\begin{aligned} 3a - b &= 1 \\ -6a + 2b &= 1. \end{aligned} \longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 1 \\ -6 & 2 & 1 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{cc|c} 3 & -1 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

system is inconsistent.

There is no solution!!!