Section 2.3: Properties of Determinants; Cramer's Rule

Objectives.

- Understand how determinants interact with matrix operations.
- Introduce the adjoint of a square matrix.
- Apply Cramer's Rule to solve a linear system.

We have several methods for finding the determinant of a matrix. We now want to find ways to deal with determinants of expressions such as kA, $A + B$, AB, and A^{-1}.

If A is an $n \times n$ matrix, and k is a scalar, then $\det(kA) = k^n \det A$.

Example 1. Confirm the property above for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the scalar k.

$$\det(kA) = \det\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = (ka)(kd) - (kb)(kc)$$

$$= k^2(ad - bc) = k^2 \det A.$$

If A and B are square matrices of the same size, then $\det(AB) = (\det A)(\det B)$.

Example 2. Confirm the property above for the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$.

$\det A = 4 - (-4) = 8$, $\det B = 6 - 5 = 1$, $(\det A)(\det B) = 8$.

equal!!!

$AB = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \cancel{-6} \; -7 & 12 \\ -10 & 16 \end{bmatrix}$, $\det(AB) = -112 - (-120) = 8$.

If A is an invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

Example 3. Suppose that A is invertible. Use $\det(AB) = (\det A)(\det B)$ to prove that $\det(A^{-1}) = \dfrac{1}{\det A}$.

If A is invertible, then A^{-1} exists and

$\det I = \det(AA^{-1}) = (\det A)(\det A^{-1})$, so $1 = (\det A)(\det A^{-1})$.

Therefore, $\det(A^{-1}) = \dfrac{1}{\det A}$.

For most pairs of matrices, the determinant of the sum is **not** the sum of the determinants.

In general, $\det(A+B) \neq \det A + \det B$.

Example 4. Confirm the property above for the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$.

$\det A = 8$, $\det B = 1$, $\det A + \det B = 9$.

$A+B = \begin{bmatrix} -1 & 4 \\ 5 & 0 \end{bmatrix}$, $\det(A+B) = -20$.

not equal!!!

The one situation where the sum of two determinants is useful is when two matrices are almost identical.

Theorem. Let A, B, and C be square matrices that differ only in row i, and suppose that the ith row of C is the sum of the ith row of A and the ith row of B. Then $\det C = \det A + \det B$.

Why? cofactor expansion!!!

cofactor exp. along row i.

$$\det C = c_{i1}C_{i1} + c_{i2}C_{i2} + \cdots + c_{in}C_{in} = (a_{i1}+b_{i1})C_{i1} + (a_{i2}+b_{i2})C_{i2} + \cdots + (\cdots)$$

$$= a_{i1}C_{i1} + \cdots + a_{in}C_{in} + b_{i1}C_{i1} + \cdots + b_{in}C_{in} = \det A + \det B.$$

A, B, C have the same cofactors along row i.

Example 5. Confirm this theorem for the matrices $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

$$\det A = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -1 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} = 2(-3-4) = -14.$$

$$\det B = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 3(2-4) = -6.$$

$$\det C = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 2 & 0 \end{vmatrix} \xrightarrow{R_3 \to R_3 - R_2} = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -2 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = 2(-6-4) = -20.$$

Thus $\det C = \det A + \det B$.

The (classical) adjoint of a square matrix A is formed by transposing the matrix of cofactors.

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Example 6. Find the adjoint of $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix}$.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} = -2 \qquad C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} = -(-6) = 6 \qquad C_{13} = \cdots = -6$$

$$C_{21} = \cdots = 1 \qquad C_{22} = \cdots = -3 \qquad C_{23} = 0$$

$$C_{31} = 0 \qquad C_{32} = -6 \qquad C_{33} = 6.$$

$$\operatorname{adj} A = \begin{bmatrix} -2 & 6 & -6 \\ 1 & -3 & 0 \\ 0 & -6 & 6 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -6 \\ -6 & 0 & 6 \end{bmatrix}.$$

A useful application of the adjoint matrix is finding an inverse.

Theorem. If A is an invertible matrix, then $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$.

from Ex. 6:

$$A(\operatorname{adj} A) = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -6 \\ -6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

Example 7. Find the inverse of the matrix A in the previous example.

$$\det A = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{vmatrix} = -6$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{-6} \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -6 \\ -6 & 0 & 6 \end{bmatrix} = \underline{\begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & 0 \\ -1 & \frac{1}{2} & 1 \\ 1 & 0 & -1 \end{bmatrix}}.$$

Cramer's Rule. If A is an $n \times n$ matrix such that $\det A \neq 0$, then the system $A\vec{x} = \vec{b}$ has the unique solution

$$x_1 = \frac{\det A_1}{\det A}, \qquad x_2 = \frac{\det A_2}{\det A}, \qquad \ldots, \qquad x_n = \frac{\det A_n}{\det A},$$

where A_j is obtained by replacing column j of A with the vector $\vec{b}$.

Example 8. Use Cramer's Rule to solve the linear system:

$$\begin{aligned} x_1 \qquad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

$C_3 \to C_3 - 2C_1$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \det A = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -3 & 4 & 12 \\ -1 & -2 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 12 \\ -2 & 5 \end{vmatrix} = 20 - (-24) = 44.$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, \quad \det A_1 = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = \cdots = -40.$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad \det A_2 = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = \cdots = 72.$$

$$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}, \quad \det A_3 = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = \cdots = 152.$$

The solution is:

$$x_1 = \frac{\det A_1}{\det A} = \frac{-40}{44} = -\frac{10}{11}.$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{72}{44} = \frac{18}{11}.$$

$$x_3 = \frac{\det A_3}{\det A} = \frac{152}{44} = \frac{38}{11}.$$