Section 2.2: Evaluating Determinants by Row Reduction

Objectives.

- Understand how elementary row operations affect determinants.
- Use row reduction to compute determinants.
- Introduce column operations and apply them to compute determinants.

The "cofactor expansion" method for finding determinants leads to some useful observations.

Theorem. Let A be a square matrix. If A has a row (or column) of zeros, then $\det A = 0$.

eg. $\det\left(\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}\right) = 0$, $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0.$ ← notation means "determinant"

Theorem. Let A be a square matrix. Then $\det A = \det A^T$.

why? cofactor expansion on the i^{th} row of A is the same as cofactor on the i^{th} column of A^T.

Theorem. Let A be a square matrix.

(a) If B is obtained by multiplying a row (or column) of A by a scalar k, then $\det B = k \det A$.

eg. $$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(b) If B is obtained by swapping two rows (or columns) of A, then $\det B = -\det A$.

eg. $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} b & a \\ d & c \end{vmatrix}.$$

(c) If B is obtained by adding a multiple of one row of A to another (or a multiple of one column of A to another), then $\det B = \det A$.

eg. $$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Theorem. Let E be an $n \times n$ elementary matrix.

(a) If E is obtained by multiplying a row of I_n by a scalar k, then $\det E = k$.

(b) If E is obtained by swapping two rows of I_n, then $\det E = -1$.

(c) If E is obtained by adding a multiple of one row of I_n to another, then $\det E = 1$.

eg. $\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = k$, $\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = -1$, $\det\left(\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1$.

Theorem. Let A be a square matrix. If two rows (or two columns) of A are proportional, then $\det A = 0$.

eg. $\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = 0$, $\det\left(\begin{bmatrix} 1 & 3 & -1 \\ 3 & 9 & -3 \\ 3 & 0 & 4 \end{bmatrix}\right) = 0$

$C_2 = 2C_1$ $R_2 = 3R_1$

Example 1. Find each determinant.

(a) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_3} = - \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -2.$ ← diagonal matrix!!!

(b) $\begin{vmatrix} 1 & 0 & -4 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} = -3.$ ← upper triangular!!! [alternative: $R_1 \to R_1 + 4R_3$].

$R_2 \to R_2 - R_1$

(c) $\begin{vmatrix} 1 & 7 & 3 & 0 & 2 \\ 0 & -1 & -5 & 0 & 0 \\ -1 & 2 & -2 & 0 & -2 \\ 3 & 0 & 5 & 1 & 6 \\ 1 & 0 & 0 & 0 & 2 \end{vmatrix} = 0$, because $C_5 = 2C_1$

Example 2. Use row reduction to compute each determinant.

(a) $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$ • swap R_1 and R_2 → multiply by -1

$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$ $R_1 \to \frac{1}{3}R_1$ → take factor of 3 outside the determinant.

$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$ $R_3 \to R_3 - 2R_1$ → determinant does not change!!!

$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$ $R_3 \to R_3 - 10R_2$ → determinant does not change!!!

$= -3(1)(1)(-55)$

$= \underline{165}.$

(b) $\begin{vmatrix} -1 & 4 & 2 & 6 \\ 0 & 0 & 1 & 7 \\ -1 & 2 & 4 & 14 \\ 0 & 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 4 & 2 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & -2 & 2 & 8 \\ 0 & 2 & 4 & 6 \end{vmatrix}$ $R_3 \to R_3 - R_1$

cofactor expansion along column 1.

$= (-1)\begin{vmatrix} 0 & 1 & 7 \\ -2 & 2 & 8 \\ 2 & 4 & 6 \end{vmatrix}$

$R_2 \to R_2 + R_3$

$= -\begin{vmatrix} 0 & 1 & 7 \\ 0 & 6 & 14 \\ 2 & 4 & 6 \end{vmatrix}$ cofactor expansion along column 1.

$= -2\begin{vmatrix} 1 & 7 \\ 6 & 14 \end{vmatrix}$

$= -2(14 - 42)$

$= \underline{56}.$

We can also use column operations to simplify determinant calculations.

Example 3. Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 7 & 0 & -4 \\ 1 & -3 & 3 & 2 \\ 2 & 6 & -5 & 3 \end{bmatrix}$

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -2 & 7 & 0 & 4 \\ 1 & -3 & 3 & 2 \\ 2 & 6 & -5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 5 & 0 & 0 \\ 1 & -2 & 3 & 0 \\ 2 & 8 & -5 & -1 \end{vmatrix}$$

$C_2 \to C_2 + C_1$
$C_4 \to C_4 - 2C_1$

$= (1)(5)(3)(-1)$

$= -15.$

(b) $B = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$

$$\det(B) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix}$$

$R_1 \to R_1 - 3R_2$
$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 3R_2$

$$= \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

cofactor expansion!!!

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$R_3 \to R_3 + R_1$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

cofactor expansion!!!

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$

$= -18.$

Determinants and Solutions of Linear Systems.

In Sections 1.5 and 1.6, we learned about the "Equivalance Theorem", which gives several conditions that are equivalent to a linear system having a unique solution. We can now add a condition involving determinants.

Equivalence Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.

Section 1.5

5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.

Section 1.6

7. $\det A \neq 0$

Section 2.3.

Example 4. Which of the following matrices is invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & -5 \\ 2 & 0 & 2 \end{bmatrix}$$

$\det A = 0$, so A is not invertible

$\det B \neq 0$, so B is invertible

$\det C \neq 0$, so C is invertible (eg. swap R_1 and R_2)

$\det D = 0$ ($R_3 = 2R_1$) so D is not invertible!!!

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 & 1 & 5 \\ -4 & 0 & 4 & 1 \\ 0 & 0 & 6 & 2 \\ 2 & 0 & -3 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

can make triangular by swapping rows
$\det F \neq 0$
$\Rightarrow F$ is invertible.

$\det G = 0$
(column of zeros)
$\Rightarrow G$ is not invertible.

$R_3 = R_1 + R_2$,
so $\det H = 0$
$\Rightarrow H$ is not invertible