Section 2.1: Determinants by Cofactor Expansion

Objectives.

- Understand how to find minors and cofactors.
- Use minors and cofactors to compute the determinant of a square matrix.
- Find the determinant of a 3×3 matrix efficiently.

Recall that the determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

notation: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

We will use this to *inductively/recursively* define determinants for larger square matrices.

If $A = [a_{ij}]$ is a square matrix, then

- the minor of a_{ij} is the determinant of the submatrix obtained from A by deleting row i and column j.

  M_{ij}

- the cofactor of a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

  C_{ij}

Example 1. Let $A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \\ -1 & 8 & 2 \end{bmatrix}$.

(a) Find the minor of a_{11} and the cofactor of a_{11}.

$$M_{11} = \det\left(\begin{bmatrix} 3 & 5 \\ 8 & 2 \end{bmatrix}\right) = 6 - 40 = \underline{-34}.$$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2(-34) = \underline{-34}.$$

(b) Find the minor of a_{23} and the cofactor of a_{23}.

$$M_{23} = \begin{vmatrix} 2 & -1 \\ -1 & 8 \end{vmatrix} = 16 - 1 = \underline{15}.$$

$$C_{23} = (-1)^{2+3} M_{23} = (-1)^5(15) = \underline{-15}.$$

Cofactor Expansion.

If A is an $n \times n$ matrix, then the determinant of A is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad \text{expansion along } i^{th} \text{ row.}$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad \text{expansion along } j^{th} \text{ column.}$$

Example 2. Write out the cofactor expansion of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ along the first column.

$$\det(A) = \underset{a}{a_{11}}\,\underset{(-1)^2 d}{C_{11}} + \underset{c}{a_{21}}\,\underset{(-1)^3 b}{C_{21}} = ad + c(-b) = \underline{ad - bc.}$$

Example 3. Find the determinant of the matrix $B = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{bmatrix} \end{array}$.

$$\det(B) = 1\begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & -2 \\ 4 & 5 \end{vmatrix}$$

$$= \big((-2)(2) - (3)(5)\big) - 3\big((2)(2) - (3)(4)\big) + 0\big((2)(5) - (-2)(4)\big)$$

$$= -19 - 3(-8) + 0 = \underline{5.}$$

lots of zeros!!!

Example 4. Find the determinant of the matrix $C = \begin{array}{c} \begin{array}{cccc} + & - & + & \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 1 & 0 & 5 & 2 \\ -1 & 1 & 0 & 3 \end{bmatrix} \end{array}$.

$$\det(A) = 0\begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 & 4 \\ 0 & 1 & -3 \\ -1 & 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 & 4 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 5\left(2\begin{vmatrix} 1 & -3 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} -1 & 4 \\ 1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix}\right)$$

$$= 5\big(2(3+3) - 0 - (3-4)\big)$$

$$= \underline{65.}$$

Theorem. The determinant of an upper triangular matrix, a lower triangular matrix, or a diagonal matrix is the product of the diagonal entries.

Example 5. Show that the theorem above holds for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$.

- use cofactor expansion along column 1.

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{vmatrix} - 0 + 0 - 0$$

$$= a_{11}\left(a_{22} \begin{vmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{vmatrix} - 0 + 0\right)$$

$$= \underline{a_{11} a_{22} a_{33} a_{44}.}$$

Finding determinants can be very time-consuming, especially for large matrices. There is an efficient method for computing the determinant of a 3×3 matrix (without using cofactor expansion) that is similar to how we compute the determinant of a 2×2 matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$-bc \qquad +ad$

Example 6. Find the determinant of $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{bmatrix}$.

$$\left|\begin{matrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{matrix}\right| \begin{matrix} 1 & 3 \\ 2 & -2 \\ 4 & 5 \end{matrix}$$

$-0 \quad -15 \quad -12 \quad +(-4) \quad +36 \quad +0$

$$\det(B) = [-4 + 36 + 0] + [-0 - 15 - 12]$$

$$= \underline{5.}$$

Example 7. Find all values of λ for which the determinant of $A = \begin{bmatrix} \lambda+1 & 1 \\ 4 & \lambda-2 \end{bmatrix}$ is 0.

$$\det(A) = (\lambda+1)(\lambda-2) - 4 = \lambda^2 - \lambda - 2 - 4 = \lambda^2 - \lambda - 6 = (\lambda+2)(\lambda-3).$$

Thus $\det(A) = 0$ if $\lambda = -2$ or $\lambda = 3$.

So ... what is a determinant?

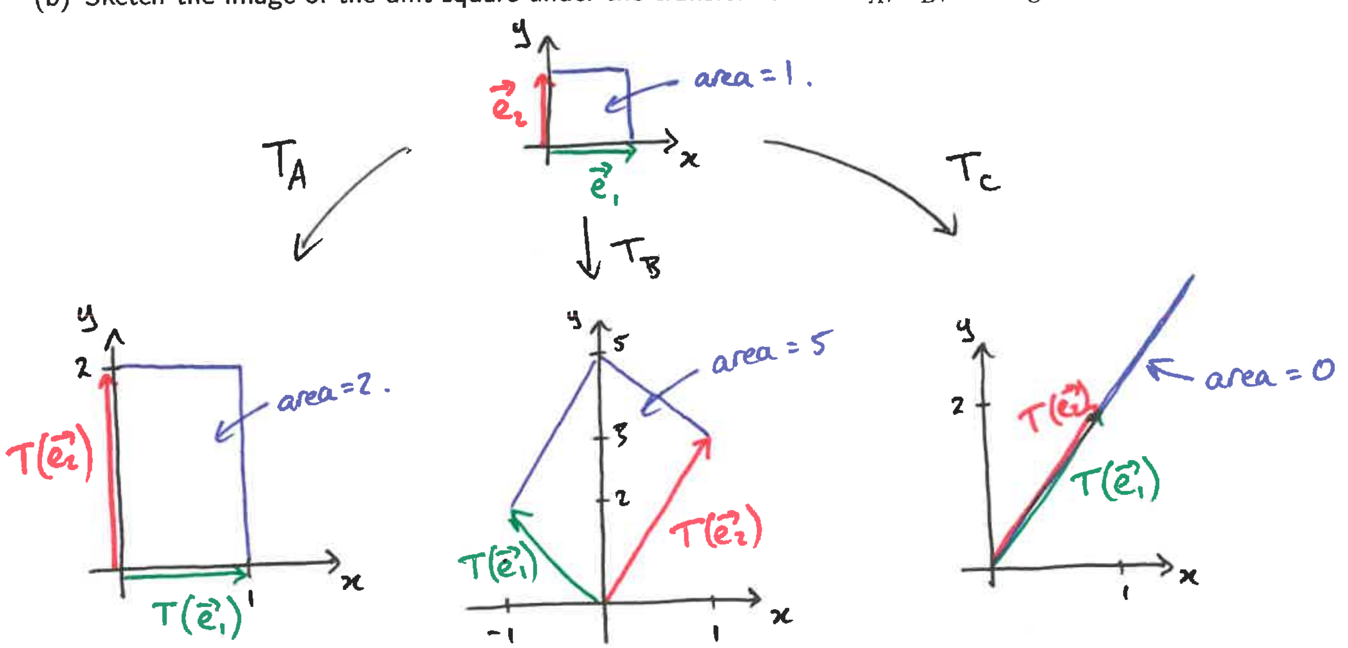
In some sense, the determinant of a square matrix A is a scaling factor for the linear transformation T_A. For instance, if A is a 2×2 matrix, then (the absolute value of) $\det A$ is the area of the parallelogram obtained by applying T_A to the unit square.

Example 8. Consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Find $\det A$, $\det B$, and $\det C$.

$\det(A) = 2$, $\det(B) = -5$, $\det(C) = 0$.

(b) Sketch the image of the unit square under the transformations T_A, T_B, and T_C.



(c) Compare the determinants in part (a) with each image in part (b).