Section 3.3: Orthogonal projections in $\mathbb{R}^3$

The orthogonal projections of a vector $\vec{x} = (x, y, z)$ in $\mathbb{R}^3$ onto each of the coordinate axes are given by:

$$T_x(\vec{x}) = (x, 0, 0) \qquad \textit{projection onto } x\textit{-axis,}$$
$$T_y(\vec{x}) = (0, y, 0) \qquad \textit{projection onto } y\textit{-axis,}$$
$$T_z(\vec{x}) = (0, 0, z) \qquad \textit{projection onto } z\textit{-axis.}$$

Problem 1. Let $\vec{x} = (x, y, z)$ be a vector in $\mathbb{R}^3$.

(a) Show that the vectors $T_x(\vec{x})$ and $T_y(\vec{x})$ are orthogonal.

$$\begin{aligned} T_x(\vec{x}) \cdot T_y(\vec{x}) &= (x, 0, 0) \cdot (0, y, 0) \\ &= x \cdot 0 + 0 \cdot y + 0 \cdot 0 \\ &= 0. \end{aligned}$$

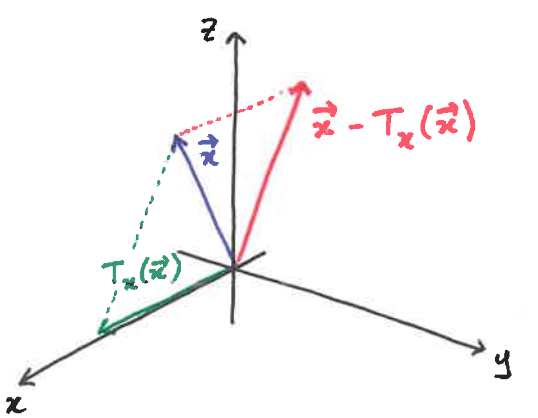
Thus $T_x(\vec{x})$ and $T_y(\vec{x})$ are orthogonal.

(b) Show that the vectors $T_x(\vec{x})$ and $\vec{x} - T_x(\vec{x})$ are orthogonal.

$$\begin{aligned} T_x(\vec{x}) \cdot (\vec{x} - T_x(\vec{x})) &= (x, 0, 0) \cdot ((x, y, z) - (x, 0, 0)) \\ &= (x, 0, 0) \cdot (0, y, z) \\ &= 0. \end{aligned}$$

Thus $T_x(\vec{x})$ and $\vec{x} - T_x(\vec{x})$ are orthogonal.

(c) Sketch a diagram showing $\vec{x}$, $T_x(\vec{x})$, and $\vec{x} - T_x(\vec{x})$.



Section 3.4: Transformations of lines in $\mathbb{R}^n$

Recall that a line in $\mathbb{R}^n$ can be represented by the equation

$$\vec{x} = \vec{x}_0 + t\vec{v},$$

where $\vec{x}$ is a general point on the line, $\vec{x}_0$ is a fixed point on the line, and $\vec{v}$ is a nonzero vector parallel to the line.

Problem 2. Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator, so that A is an invertible $n \times n$ matrix.

(a) Show that the image of the line $\vec{x} = \vec{x}_0 + t\vec{v}$ in $\mathbb{R}^n$ under the transformation T_A is also a line in $\mathbb{R}^n$.

$$\begin{aligned} T_A(\vec{x}) &= T_A(\vec{x}_0 + t\vec{v}) \\ &= T_A(\vec{x}_0) + t\,T_A(\vec{v}) \\ &= A\vec{x}_0 + tA\vec{v}. \end{aligned}$$

Because $A\vec{x}_0$ is a vector in $\mathbb{R}^n$, and $A\vec{v}$ is a nonzero vector in $\mathbb{R}^n$ (since A is invertible), this represents a line in $\mathbb{R}^n$.

(b) Let $A = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$. Find vector and parametric equations for the image of the line $\vec{x} = (1,3) + t(2,-1)$ under multiplication by A.

$$A\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}5\\-9\end{bmatrix}, \quad A\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}3\\10\end{bmatrix}$$

The image of the line $\vec{x} = (1,3) + t(2,-1)$ is the line $\vec{x} = (5,-9) + t(3,10)$.

The parametric equations are $x = 5 + 3t$ and $y = -9 + 10t$.

Section 10.1: Constructing Curves and Surfaces Through Specified Points

Lines in $\mathbb{R}^2$

Any two distinct points (x_1, y_1), (x_2, y_2) in $\mathbb{R}^2$ lie a (unique) line $c_1x + c_2y + c_3 = 0$, where at least one of c_1 and c_2 is not zero. This implies that the homogeneous linear system

$$\begin{aligned} xc_1 + \ \ yc_2 + c_3 &= 0 \\ x_1c_1 + y_1c_2 + c_3 &= 0 \\ x_2c_1 + y_2c_2 + c_3 &= 0 \end{aligned}$$

has a non-trivial solution; equivalently the determinant of the coefficient matrix is zero, which gives the following equation for the line through (x_1, y_1) and (x_2, y_2).

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Problem 3. Consider the line in $\mathbb{R}^2$ through the two points $(3, 1)$ and $(5, -8)$.

(a) Use the determinant above to find an equation for the line.

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 5 & -8 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad x\begin{vmatrix} 1 & 1 \\ -8 & 1 \end{vmatrix} - y\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 5 & -8 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(1+8) - y(3-5) + (-24-5) = 0$$

$$\Rightarrow \quad \underline{9x + 2y - 29 = 0.}$$

(b) Find the points where the line intersects each of the coordinate axes.

If $y = 0$ then $x = \frac{29}{9}$. If $x = 0$ then $y = \frac{29}{2}$.

The line intersects the axes at $\left(\frac{29}{9}, 0\right)$ and $\left(0, \frac{29}{2}\right)$.

(c) Graph the equation from part (a) to confirm that the line passes through the two given points.

Circles in $\mathbb{R}^2$

The same method can be used to find a determinant equation for the unique circle

$$c_1(x^2+y^2)+c_2x+c_3y+c_4=0$$

through three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) not on the same line.

Problem 4. Suppose the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) all lie on the circle $c_1(x^2+y^2)+c_2x+c_3y+c_4=0$.

(a) Set up a homogeneous system of linear equations in c_1, c_2, c_3, and c_4 satisfied by the three given points and a general point (x, y) on the same circle.

$$\begin{aligned}
c_1(x^2+y^2)+c_2x+c_3y+c_4&=0\\
c_1(x_1^2+y_1^2)+c_2x_1+c_3y_1+c_4&=0\\
c_1(x_2^2+y_2^2)+c_2x_2+c_3y_2+c_4&=0\\
c_1(x_3^2+y_3^2)+c_2x_3+c_3y_3+c_4&=0
\end{aligned}$$

(b) The system in part (a) has non-trivial solutions. Write a determinant equation to represent this.

$$\begin{vmatrix} x^2+y^2 & x & y & 1\\ x_1^2+y_1^2 & x_1 & y_1 & 1\\ x_2^2+y_2^2 & x_2 & y_2 & 1\\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0$$

(c) Find the center and the radius of the circle passing through $(2, -2)$, $(3, 5)$, and $(-4, 6)$.

$$\begin{vmatrix} x^2+y^2 & x & y & 1\\ 8 & 2 & -2 & 1\\ 34 & 3 & 5 & 1\\ 52 & -4 & +6 & 1 \end{vmatrix}=0$$

$$\Rightarrow\quad 50x^2+100x+50y^2-200y-1000=0$$

$$\Rightarrow\quad x^2+2x+y^2-4y=20$$

$$\Rightarrow\quad (x+1)^2+(y-2)^2=25 \qquad \text{center} = (-1, 2),\ \text{radius} = 5.$$

(d) Graph the equation from part (c) to confirm that the circle passes through the three given points.

Conic sections in $\mathbb{R}^2$

A general conic section in $\mathbb{R}^2$ has equation

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0,$$

and is determined by five distinct points in the plane.

Problem 5. (a) Find a determinant equation for the conic section through the five distinct points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5).$$

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0$$

(b) Find an equation for the conic section through the points $(0,0)$, $(0,-1)$, $(2,0)$, $(2,-5)$, and $(4,-1)$.

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 4 & 0 & 0 & 2 & 0 & 1 \\ 4 & -10 & 25 & 2 & -5 & 1 \\ 16 & -4 & 1 & 4 & -1 & 1 \end{vmatrix} = 0$$

$$\Longrightarrow \quad 160x^2 + 320xy + 320y^2 - 320x + 320y = 0$$

$$\Longrightarrow \quad x^2 + 2xy + y^2 - 2x + 2y = 0$$

(c) Graph the equation from part (b). What type of conic section is this?

a parabola!!!

Planes in $\mathbb{R}^3$

A plane in $\mathbb{R}^3$ has the scalar equation $c_1x + c_2y + c_3z + c_4 = 0$, and is determined by three points not on the same line.

Problem 6. (a) Find a determinant equation for the plane through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

(b) Find a scalar equation of the plane through the points $(2, 1, 3)$, $(2, -1, -1)$, and $(1, 1, 2)$.

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 3 & 1 \\ 2 & -1 & -1 & 1 \\ 1 & 1 & 2 & 1 \end{vmatrix} = 0 \implies 2x + 4y - 2z - 1 = 0.$$

(c) Graph the equation from part (c) to confirm that the plane passes through the three given points.

Spheres in $\mathbb{R}^3$

A sphere in $\mathbb{R}^3$ has equation

$$c_1(x^2 + y^2 + z^2) + c_2x + c_3y + c_4z + c_5 = 0,$$

and is determined by four points not in the same plane.

Problem 7. (a) Find a determinant equation for the sphere through the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4).

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

(b) Find an equation of the sphere through the points $(0, 1, -2)$, $(1, 3, 1)$, $(2, -1, 0)$, and $(3, 1, -1)$.

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 5 & 0 & 1 & -2 & 1 \\ 11 & 1 & 3 & 1 & 1 \\ 5 & 2 & -1 & 0 & 1 \\ 11 & 3 & 1 & -1 & 1 \end{vmatrix} = 0 \implies \underline{x^2 - 2x + y^2 - 2y + z^2 = 3}.$$

(c) Graph the equation from part (c) to confirm that the sphere passes through the four given points.