Section 2.1: Determinants by Cofactor Expansion

Objectives.

- Understand how to find minors and cofactors.
- Use minors and cofactors to compute the determinant of a square matrix.
- Find the determinant of a 3×3 matrix efficiently.

Recall that the determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

notation: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

We will use this to *inductively/recursively* define determinants for larger square matrices.

If $A = [a_{ij}]$ is a square matrix, then

- the minor of a_{ij} is the determinant of the submatrix obtained from A by deleting row i and column j.

  M_{ij}

- the cofactor of a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

  C_{ij}

Example 1. Let $A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \\ -1 & 8 & 2 \end{bmatrix}$.

(a) Find the minor of a_{11} and the cofactor of a_{11}.

$$M_{11} = \det\left(\begin{bmatrix} 3 & 5 \\ 8 & 2 \end{bmatrix}\right) = 6 - 40 = \underline{-34}.$$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2(-34) = \underline{-34}.$$

(b) Find the minor of a_{23} and the cofactor of a_{23}.

$$M_{23} = \begin{vmatrix} 2 & -1 \\ -1 & 8 \end{vmatrix} = 16 - 1 = \underline{15}.$$

$$C_{23} = (-1)^{2+3} M_{23} = (-1)^5(15) = \underline{-15}.$$

Cofactor Expansion.

If A is an $n \times n$ matrix, then the determinant of A is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad \text{expansion along } i^{th} \text{ row.}$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad \text{expansion along } j^{th} \text{ column.}$$

Example 2. Write out the cofactor expansion of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ along the first column.

$$\det(A) = \underbrace{a_{11}}_{a}\underbrace{C_{11}}_{(-1)^2 d} + \underbrace{a_{21}}_{c}\underbrace{C_{21}}_{(-1)^3 b} = ad + c(-b) = \underline{ad-bc.}$$

Example 3. Find the determinant of the matrix $B = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{bmatrix} \end{array}$.

$$\det(B) = 1\begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & -2 \\ 4 & 5 \end{vmatrix}$$

$$= \big((-2)(2)-(3)(5)\big) - 3\big((2)(2)-(3)(4)\big) + 0\big((2)(5)-(-2)(4)\big)$$

$$= -19 - 3(-8) + 0 \quad = \underline{5.}$$

lots of zeros!!!

Example 4. Find the determinant of the matrix $C = \begin{array}{c} \begin{array}{cccc} + & - & + & \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 1 & 0 & 5 & 2 \\ -1 & 1 & 0 & 3 \end{bmatrix} \end{array}$.

$$\det(A) = 0\begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 & 4 \\ 0 & 1 & -3 \\ -1 & 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 & 4 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 5\left(2\begin{vmatrix} 1 & -3 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} -1 & 4 \\ 1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix}\right)$$

$$= 5\big(2(3+3) - 0 - (3-4)\big)$$

$$= \underline{65.}$$

Theorem. The determinant of an upper triangular matrix, a lower triangular matrix, or a diagonal matrix is the product of the diagonal entries.

Example 5. Show that the theorem above holds for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$.

- use cofactor expansion along column 1.

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{vmatrix} - 0 + 0 - 0$$

$$= a_{11} \left(a_{22} \begin{vmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{vmatrix} - 0 + 0 \right)$$

$$= \underline{a_{11} a_{22} a_{33} a_{44}.}$$

Finding determinants can be very time-consuming, especially for large matrices. There is an efficient method for computing the determinant of a 3×3 matrix (without using cofactor expansion) that is similar to how we compute the determinant of a 2×2 matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$-bc \qquad +ad$

Example 6. Find the determinant of $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{bmatrix}$.

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & -2 \\ 4 & 5 \end{matrix}$$

$-0 \quad -15 \quad -12 \quad +(-4) \quad +36 \quad +0$

$$\det(B) = [-4 + 36 + 0] + [-0 - 15 - 12]$$

$$= \underline{5.}$$

Example 7. Find all values of λ for which the determinant of $A = \begin{bmatrix} \lambda+1 & 1 \\ 4 & \lambda-2 \end{bmatrix}$ is 0.

$$\det(A) = (\lambda+1)(\lambda-2) - 4 = \lambda^2 - \lambda - 2 - 4 = \lambda^2 - \lambda - 6 = (\lambda+2)(\lambda-3).$$

Thus $\det(A) = 0$ if $\lambda = -2$ or $\lambda = 3$.

So ... what is a determinant?

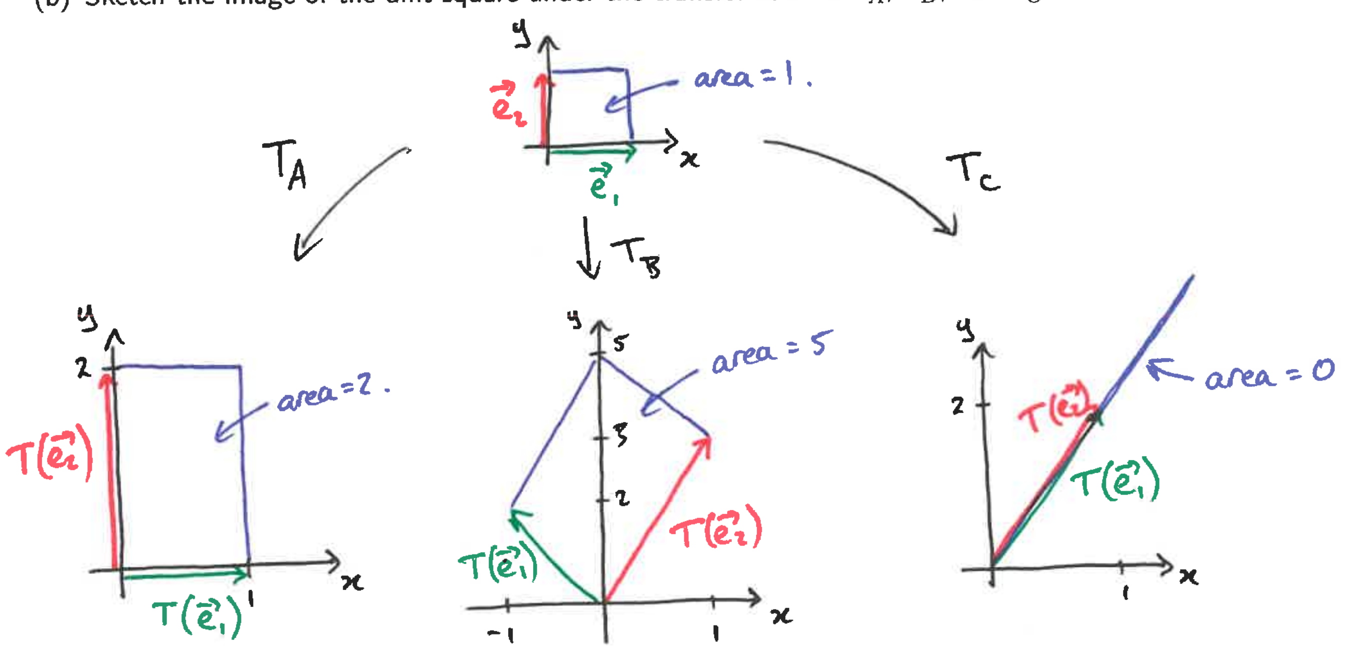
In some sense, the determinant of a square matrix A is a scaling factor for the linear transformation T_A. For instance, if A is a 2×2 matrix, then (the absolute value of) $\det A$ is the area of the parallelogram obtained by applying T_A to the unit square.

Example 8. Consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Find $\det A$, $\det B$, and $\det C$.

$\det(A) = 2$, $\det(B) = -5$, $\det(C) = 0$.

(b) Sketch the image of the unit square under the transformations T_A, T_B, and T_C.



(c) Compare the determinants in part (a) with each image in part (b).

Section 2.2: Evaluating Determinants by Row Reduction

Objectives.

- Understand how elementary row operations affect determinants.
- Use row reduction to compute determinants.
- Introduce column operations and apply them to compute determinants.

The "cofactor expansion" method for finding determinants leads to some useful observations.

Theorem. Let A be a square matrix. If A has a row (or column) of zeros, then $\det A = 0$.

eg. $\det\left(\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}\right) = 0$, $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0.$ ← notation means "determinant"

Theorem. Let A be a square matrix. Then $\det A = \det A^T$.

why? cofactor expansion on the i^{th} row of A is the same as cofactor on the i^{th} column of A^T.

Theorem. Let A be a square matrix.

(a) If B is obtained by multiplying a row (or column) of A by a scalar k, then $\det B = k \det A$.

eg. $\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(b) If B is obtained by swapping two rows (or columns) of A, then $\det B = -\det A$.

eg. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} b & a \\ d & c \end{vmatrix}.$

(c) If B is obtained by adding a multiple of one row of A to another (or a multiple of one column of A to another), then $\det B = \det A$.

eg. $\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$

Theorem. Let E be an $n \times n$ elementary matrix.

(a) If E is obtained by multiplying a row of I_n by a scalar k, then $\det E = k$.

(b) If E is obtained by swapping two rows of I_n, then $\det E = -1$.

(c) If E is obtained by adding a multiple of one row of I_n to another, then $\det E = 1$.

eg. $\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = k$, $\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = -1$, $\det\left(\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1$.

Theorem. Let A be a square matrix. If two rows (or two columns) of A are proportional, then $\det A = 0$.

eg. $\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = 0$ ($C_2 = 2C_1$), $\det\left(\begin{bmatrix} 1 & 3 & -1 \\ 3 & 9 & -3 \\ 3 & 0 & 4 \end{bmatrix}\right) = 0$ ($R_2 = 3R_1$)

Example 1. Find each determinant.

(a) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_3} = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -2.$ ← diagonal matrix!!!

(b) $\begin{vmatrix} 1 & 0 & -4 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} \xrightarrow{R_2 \to R_2 - R_1} = \begin{vmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} = -3.$ ← upper triangular!!! [alternative: $R_1 \to R_1 + 4R_3$].

(c) $\begin{vmatrix} 1 & 7 & 3 & 0 & 2 \\ 0 & -1 & -5 & 0 & 0 \\ -1 & 2 & -2 & 0 & -2 \\ 3 & 0 & 5 & 1 & 6 \\ 1 & 0 & 0 & 0 & 2 \end{vmatrix} = 0$, because $C_5 = 2C_1$

Example 2. Use row reduction to compute each determinant.

(a)
$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

• swap R_1 and R_2 → multiply by -1

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$R_1 \to \frac{1}{3}R_1$ → take factor of 3 outside the determinant.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$R_3 \to R_3 - 2R_1$ → determinant does not change!!!

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$R_3 \to R_3 - 10R_2$ → determinant does not change!!!

$$= -3(1)(1)(-55)$$
$$= 165.$$

(b)
$$\begin{vmatrix} -1 & 4 & 2 & 6 \\ 0 & 0 & 1 & 7 \\ -1 & 2 & 4 & 14 \\ 0 & 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 4 & 2 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & -2 & 2 & 8 \\ 0 & 2 & 4 & 6 \end{vmatrix}$$

$R_3 \to R_3 - R_1$

cofactor expansion along column 1.

$$= (-1)\begin{vmatrix} 0 & 1 & 7 \\ -2 & 2 & 8 \\ 2 & 4 & 6 \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= -\begin{vmatrix} 0 & 1 & 7 \\ 0 & 6 & 14 \\ 2 & 4 & 6 \end{vmatrix}$$

cofactor expansion along column 1.

$$= -2\begin{vmatrix} 1 & 7 \\ 6 & 14 \end{vmatrix}$$
$$= -2(14 - 42)$$
$$= 56.$$

We can also use column operations to simplify determinant calculations.

Example 3. Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 7 & 0 & -4 \\ 1 & -3 & 3 & 2 \\ 2 & 6 & -5 & 3 \end{bmatrix}$

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -2 & 7 & 0 & 4 \\ 1 & -3 & 3 & 2 \\ 2 & 6 & -5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 5 & 0 & 0 \\ 1 & -2 & 3 & 0 \\ 2 & 8 & -5 & -1 \end{vmatrix}$$

$C_2 \to C_2 + C_1$

$C_4 \to C_4 - 2C_1$

$= (1)(5)(3)(-1)$

$= \underline{-15}.$

(b) $B = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$

$$\det(B) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix}$$

$R_1 \to R_1 - 3R_2$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - 3R_2$

$$= \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

cofactor expansion!!!

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$R_3 \to R_3 + R_1$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

cofactor expansion!!!

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$

$= \underline{-18}.$

Determinants and Solutions of Linear Systems.

In Sections 1.5 and 1.6, we learned about the "Equivalance Theorem", which gives several conditions that are equivalent to a linear system having a unique solution. We can now add a condition involving determinants.

Equivalence Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.

Section 1.5

5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.

Section 1.6

7. $\det A \neq 0$

Section 2.3.

Example 4. Which of the following matrices is invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & -5 \\ 2 & 0 & 2 \end{bmatrix}$$

$\det A = 0$, so A is not invertible

$\det B \neq 0$, so B is invertible

$\det C \neq 0$, so C is invertible (eg. swap R_1 and R_2)

$\det D = 0$ ($R_3 = 2R_1$) so D is not invertible!!!

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 & 1 & 5 \\ -4 & 0 & 4 & 1 \\ 0 & 0 & 6 & 2 \\ 2 & 0 & -3 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

can make triangular by swapping rows
$\det F \neq 0$
$\Rightarrow F$ is invertible.

$\det G = 0$
(column of zeros)
$\Rightarrow G$ is not invertible.

$R_3 = R_1 + R_2$,
so $\det H = 0$
$\Rightarrow H$ is not invertible

Section 2.3: Properties of Determinants; Cramer's Rule

Objectives.

- Understand how determinants interact with matrix operations.
- Introduce the adjoint of a square matrix.
- Apply Cramer's Rule to solve a linear system.

We have several methods for finding the determinant of a matrix. We now want to find ways to deal with determinants of expressions such as kA, $A+B$, AB, and A^{-1}.

If A is an $n \times n$ matrix, and k is a scalar, then $\det(kA) = k^n \det A$.

Example 1. Confirm the property above for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the scalar k.

$$\det(kA) = \det\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = (ka)(kd) - (kb)(kc)$$

$$= k^2(ad-bc) = k^2 \det A.$$

If A and B are square matrices of the same size, then $\det(AB) = (\det A)(\det B)$.

Example 2. Confirm the property above for the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$.

$\det A = 4-(-4) = 8$, $\det B = 6-5 = 1$, $(\det A)(\det B) = 8$.

$AB = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -7 & 12 \\ -10 & 16 \end{bmatrix}$, $\det(AB) = -112-(-120) = 8$.

equal!!!

If A is an invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

Example 3. Suppose that A is invertible. Use $\det(AB) = (\det A)(\det B)$ to prove that $\det(A^{-1}) = \dfrac{1}{\det A}$.

If A is invertible, then A^{-1} exists and

$\det I = \det(AA^{-1}) = (\det A)(\det A^{-1})$, so $1 = (\det A)(\det A^{-1})$.

Therefore, $\det(A^{-1}) = \dfrac{1}{\det A}$.

For most pairs of matrices, the determinant of the sum is **not** the sum of the determinants.

In general, $\det(A+B) \neq \det A + \det B$.

Example 4. Confirm the property above for the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$.

$\det A = 8$, $\det B = 1$, $\det A + \det B = 9$.

$A+B = \begin{bmatrix} -1 & 4 \\ 5 & 0 \end{bmatrix}$, $\det(A+B) = -20$.

not equal!!

The one situation where the sum of two determinants is useful is when two matrices are almost identical.

Theorem. Let A, B, and C be square matrices that differ only in row i, and suppose that the ith row of C is the sum of the ith row of A and the ith row of B. Then $\det C = \det A + \det B$.

why? cofactor expansion!!!

$$\det C = \overbrace{c_{i1}C_{i1} + c_{i2}C_{i2} + \cdots + c_{in}C_{in}}^{\text{cofactor exp. along row } i} = (a_{i1}+b_{i1})C_{i1} + (a_{i2}+b_{i2})C_{i2} + \cdots + (\cdots)$$

$$= \underbrace{a_{i1}C_{i1} + \cdots + a_{in}C_{in}} + \underbrace{b_{i1}C_{i1} + \cdots + b_{in}C_{in}} = \det A + \det B.$$

A, B, C have the same cofactors along row i.

Example 5. Confirm this theorem for the matrices $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

$$\det A = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -1 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} = 2(-3-4) = -14.$$

$$\det B = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 3(2-4) = -6.$$

$$\det C = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 2 & 0 \end{vmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -2 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = 2(-6-4) = -20.$$

Thus $\det C = \det A + \det B$.

The (classical) adjoint of a square matrix A is formed by transposing the matrix of cofactors.

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Example 6. Find the adjoint of $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix}$.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} = -2 \qquad C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} = -(-6) = 6 \qquad C_{13} = \cdots = -6$$

$$C_{21} = \cdots = 1 \qquad C_{22} = \cdots = -3 \qquad C_{23} = 0$$

$$C_{31} = 0 \qquad C_{32} = -6 \qquad C_{33} = 6.$$

$$\operatorname{adj} A = \begin{bmatrix} -2 & 6 & -6 \\ 1 & -3 & 0 \\ 0 & -6 & 6 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -6 \\ -6 & 0 & 6 \end{bmatrix}.$$

A useful application of the adjoint matrix is finding an inverse.

Theorem. If A is an invertible matrix, then $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$.

from Ex. 6:

$$A(\operatorname{adj} A) = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -6 \\ -6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

Example 7. Find the inverse of the matrix A in the previous example.

$$\det A = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{vmatrix} = -6$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{-6} \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -6 \\ -6 & 0 & 6 \end{bmatrix} = \underline{\begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & 0 \\ -1 & \frac{1}{2} & 1 \\ 1 & 0 & -1 \end{bmatrix}}.$$

Cramer's Rule. If A is an $n \times n$ matrix such that $\det A \neq 0$, then the system $A\vec{x} = \vec{b}$ has the unique solution

$$x_1 = \frac{\det A_1}{\det A}, \qquad x_2 = \frac{\det A_2}{\det A}, \qquad \ldots, \qquad x_n = \frac{\det A_n}{\det A},$$

where A_j is obtained by replacing column j of A with the vector $\vec{b}$.

Example 8. Use Cramer's Rule to solve the linear system:

$$\begin{aligned} x_1 \qquad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

$C_3 \to C_3 - 2C_1$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \det A = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -3 & 4 & 12 \\ -1 & -2 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 12 \\ -2 & 5 \end{vmatrix} = 20 - (-24) = 44.$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, \quad \det A_1 = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = \cdots = -40.$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad \det A_2 = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = \cdots = 72.$$

$$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}, \quad \det A_3 = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = \cdots = 152.$$

The solution is:

$$x_1 = \frac{\det A_1}{\det A} = \frac{-40}{44} = -\frac{10}{11}.$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{72}{44} = \frac{18}{11}.$$

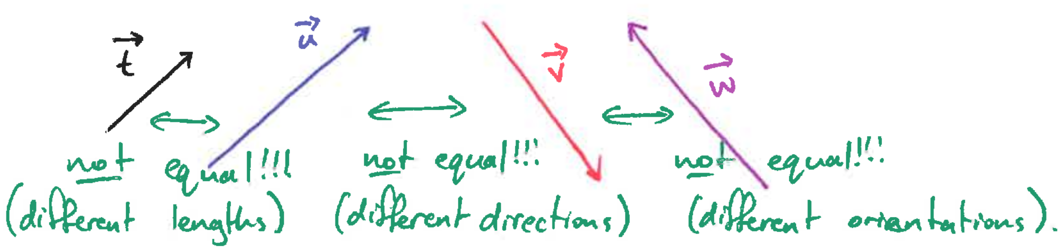
$$x_3 = \frac{\det A_3}{\det A} = \frac{152}{44} = \frac{38}{11}.$$

Section 3.1: Vectors in 2-space, 3-space, and n-space

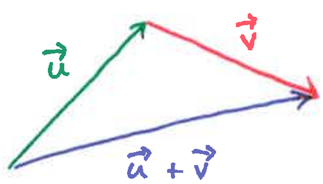
Objectives.

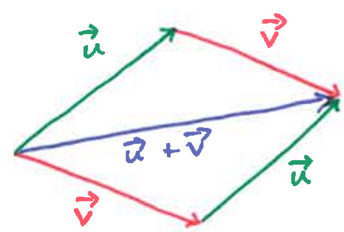
- Introduce the some terminology and notation for vectors.
- Understand vector operations in $\mathbb{R}^n$ geometrically and algebraically.
- Study some properties of vector operations.

A (geometric) vector is a quantity with a direction and a length, often represented by an arrow.



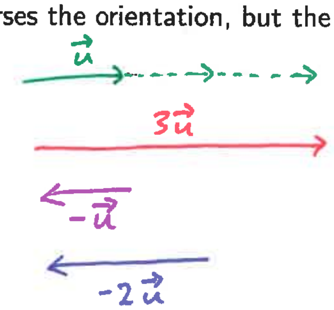
Two vectors can be added (geometrically) by placing the vectors end-to-end. (This is referred to as either the "triangle rule" or the "parallelogram rule".)





note: $\vec{u}+\vec{v}$ is the diagonal of a parallelogram with sides $\vec{u}$ and $\vec{v}$.

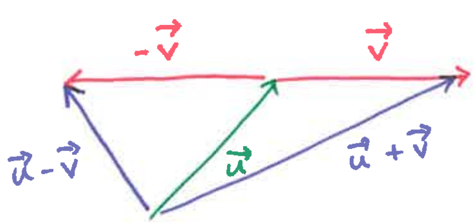
Multiplying a vector by a scalar changes ("scales") the length of the vector without changing the direction. If one vector is a scalar multiple of another, then we say the vectors are parallel. (Multiplying by a negative scalar reverses the orientation, but the result is still parallel to the original vector.)



$\vec{u}, 3\vec{u}, -\vec{u}, -2\vec{u}$ are all parallel.

note: the zero vector $\vec{0}$ is "parallel to" every vector!!!

We can view subtraction of a vector as "adding the negative of the vector".



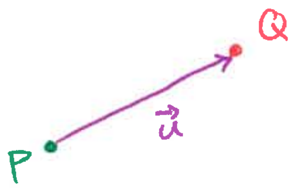
If $\underbrace{P = (a_1, a_2, \ldots, a_n)}_{\text{initial point}}$ and $\underbrace{Q = (b_1, b_2, \ldots, b_n)}_{\text{terminal point}}$ are two points in $\mathbb{R}^n$, then the vector from P to Q is

$$\overrightarrow{PQ} = (b_1 - a_1, b_2 - a_2, \cdots, b_n - a_n).$$

Two vectors $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ are equal if their components are equal. That is:

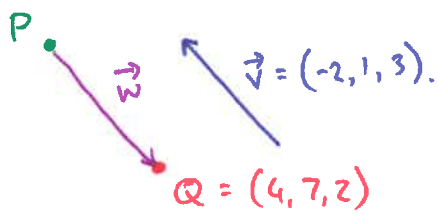
$$\vec{u} = \vec{v} \iff u_1 = v_1 \text{ and } u_2 = v_2 \text{ and } \cdots \text{ and } u_n = v_n.$$

Example 1. Find the vector $\vec{u} = \overrightarrow{PQ}$ that has initial point $P = (3, -1)$ and terminal point $Q = (-2, 8)$.



$$\vec{u} = \overrightarrow{PQ} = (-2-3, 8-(-1)) = \underline{(-5, 9)}.$$

Example 2. Find the initial point of a vector $\vec{w}$ that has terminal point $Q = (4, 7, 2)$ and is parallel to $\vec{v} = (-2, 1, 3)$ but has the opposite orientation. i.e. choose $\vec{w} = k\vec{v}$ where $k < 0$.



$$P = (4 + (-2), 7+1, 2+3) = \underline{(2, 8, 5)}.$$

Arithmetic with vectors (addition, subtraction, scalar multiplication) is done componentwise. If $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$ and k is a scalar, then we define:

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$
$$k\vec{u} = (ku_1, ku_2, \ldots, ku_n)$$
$$-\vec{u} = (-u_1, -u_2, \cdots, -u_n)$$

Example 3. Let $\vec{u} = (3, 1, 4, -2)$ and $\vec{v} = (1, -2, 3, 0)$. Simplify:

(a) $\vec{u} + \vec{v} = (3,1,4,-2) + (1,-2,3,0) = \underline{(4,-1,7,-2)}$.

(b) $3\vec{u} - 4\vec{v} = 3(3,1,4,-2) - 4(1,-2,3,0) = (9,3,12,-6) - (4,-8,12,0)$

$= \underline{(5, 11, 0, -6)}$.

Properties of vector operations. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and k and m are scalars, then:

1. $(\vec{u}+\vec{v})+\vec{w} = \vec{u} + (\vec{v}+\vec{w})$
2. $\vec{u}+\vec{v} = \vec{v} + \vec{u}$
3. $\vec{u}+\vec{0} = \vec{u}$
4. $\vec{u}+(-\vec{u}) = \vec{0}$
5. $k(\vec{u}+\vec{v}) = k\vec{u} + k\vec{v}$
6. $(k+m)\vec{u} = k\vec{u} + m\vec{u}$
7. $k(m\vec{u}) = (km)\vec{u} = m(k\vec{u})$
8. $1\vec{u} = \vec{u}$

Proof of 2. Let $\vec{u} = (u_1, u_2, \cdots, u_n)$ and $\vec{v} = (v_1, v_2, \cdots, v_n)$. Then

$$\begin{aligned}\vec{u}+\vec{v} &= (u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) && \text{def. of vector addition}\\ &= (u_1+v_1, u_2+v_2, \cdots, u_n+v_n) && \text{addition in } \mathbb{R} \text{ is commutative.}\\ &= (v_1+u_1, v_2+u_2, \cdots, v_n+u_n) && \text{def. of vector addition.}\\ &= (v_1, v_2, \cdots, v_n) + (u_1, u_2, \cdots, u_n)\\ &= \vec{v}+\vec{u}.\end{aligned}$$

Example 4. Let $\vec{u} = (-1, 4, 6)$ and $\vec{v} = (3, 3, 3)$. Find the vector $\vec{x}$ satisfying $4\vec{x} - 2\vec{u} = 2\vec{x} - \vec{v}$.

$$4\vec{x} - 2\vec{u} = 2\vec{x} - \vec{v} \implies 2\vec{x} = 2\vec{u} - \vec{v}$$

$$\implies \vec{x} = \tfrac{1}{2}(2\vec{u} - \vec{v}) = \vec{u} - \tfrac{1}{2}\vec{v}$$

$$= (-1, 4, 6) - \tfrac{1}{2}(3,3,3)$$

$$= \left(-\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{9}{2}\right).$$

zero scalar zero vector

Theorem. If $\vec{v}$ is a vector in $\mathbb{R}^n$ and k is a scalar, then

1. $0\vec{v} = \vec{0}$
2. $k\vec{0} = \vec{0}$
3. $(-1)\vec{v} = -\vec{v}$

Proof of 1. Let $\vec{v} = (v_1, v_2, \cdots, v_n)$. Then:

$$0\vec{v} = 0(v_1, v_2, \cdots, v_n) = (0v_1, 0v_2, \cdots, 0v_n) = (0, 0, \cdots, 0) = \vec{0}.$$

A vector $\vec{w}$ in $\mathbb{R}^n$ is a <u>linear combination</u> of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r} \in \mathbb{R}^n$ if

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r$$, where $k_1, k_2, \ldots, k_r$ are scalars.

Example 5. Find scalars c_1, c_2, c_3 satisfying $c_1(1,2,2) + c_2(0,1,-1) + c_3(3,1,2) = (-1,7,7)$.

- i.e. write $(-1,7,7)$ as a linear combination of $(1,2,2)$, $(0,1,-1)$, $(3,1,2)$.

This equation is equivalent to the linear system

$$\begin{aligned} c_1 \quad\;\; + 3c_3 &= -1 \\ 2c_1 + c_2 + c_3 &= 7 \\ 2c_1 - c_2 + 2c_3 &= 7 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 2 & 1 & 1 & 7 \\ 2 & -1 & 2 & 7 \end{array}\right].$$

We can reduce this to rref using Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]. \leftarrow \text{rref for linear system.}$$

That is, $c_1 = 5$, $c_2 = -1$, $c_3 = -2$.

Example 6. Show that there is no choice of scalars a and b such that $a(3,-6) + b(-1,2) = (1,1)$.

We need to solve the system

$$\begin{aligned} 3a - b &= 1 \\ -6a + 2b &= 1. \end{aligned} \longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 1 \\ -6 & 2 & 1 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{cc|c} 3 & -1 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

system is inconsistent.

There is no solution!!!

Section 3.2: Norm, Dot Product, and Distance in $\mathbb{R}^n$

Objectives.

- Define and apply the notions of norm and distance in $\mathbb{R}^n$.
- Introduce the dot product of two vectors, and interpret the dot product geometrically.
- Study some properties and applications of the dot product.

The norm (length, magnitude) of a vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

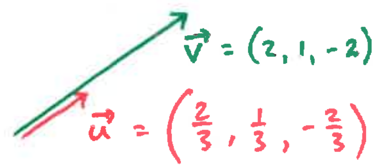
note: this generalizes Pythagoras!!!

Dividing a (non-zero) vector $\vec{v}$ by its norm produces the unit vector in the same direction as $\vec{v}$.

Example 1. Find the unit vector $\vec{u}$ that has the same direction as $\vec{v} = (2, 1, -2)$. Check that $\|\vec{u}\| = 1$.

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3.$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{3}(2,1,-2) = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right).$$



check: $\|\vec{u}\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{9}{9}} = 1.$

The distance between two points $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Example 2. Find the distance between the points $\vec{u} = (1, 3, -2, 0, 2)$ and $\vec{v} = (3, 0, 1, 1, -1)$ in $\mathbb{R}^5$.

$$d(\vec{u}, \vec{v}) = \sqrt{(1-3)^2 + (3-0)^2 + (-2-1)^2 + (0-1)^2 + (2-(-1))^2}$$

$$= \sqrt{4 + 9 + 9 + 1 + 9}$$

$$= \sqrt{32}$$

$$= 4\sqrt{2}.$$

The dot product of two vectors $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

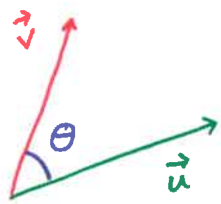
$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

note: vector · vector = scalar.

Example 3. Find the dot product of the vectors $\vec{u} = (1, 3, 2, 4)$ and $\vec{v} = (-1, 1, -2, 1)$

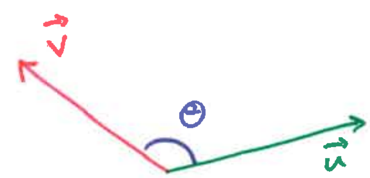
$$\begin{aligned}\vec{u}\cdot\vec{v} &= (1,3,2,4)\cdot(-1,1,-2,1)\\ &= -1 + 3 - 4 + 4\\ &= \underline{2}.\end{aligned}$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the dot product of two vectors is related to the angle between them. (This can also be generalized to finding "angles" between vectors in higher-dimensional spaces.)



$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

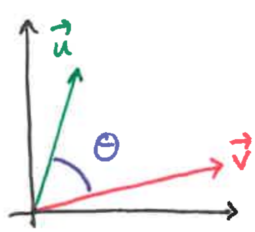
$$\Rightarrow \cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$



θ acute $\Leftrightarrow$ $\vec{u}\cdot\vec{v} > 0$.

θ obtuse $\Leftrightarrow$ $\vec{u}\cdot\vec{v} < 0$.

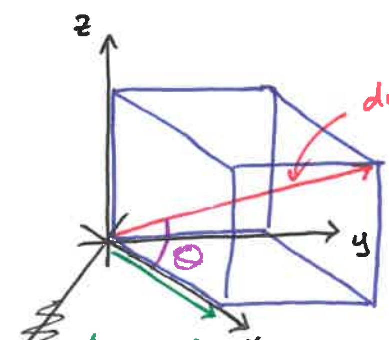
Example 4. Find the angle between the vectors $\vec{u} = (1, 2)$ and $\vec{v} = (3, 1)$.



$$\vec{u}\cdot\vec{v} = 5, \quad \|\vec{u}\| = \sqrt{5}, \quad \|\vec{v}\| = \sqrt{10}$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}} \Rightarrow \underline{\theta = 45^\circ}.$$

Example 5. Find the angle between a diagonal and an edge of a cube.



$$\cos\theta = \frac{(1,1,1)\cdot(1,0,0)}{\|(1,1,1)\|\,\|(1,0,0)\|} = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \underline{54.74^\circ}.$$

Notice that the dot product of a vector with itself is the square of the norm of the vector.

If $\vec{v} = (v_1, v_2, \dots, v_n)$ is a vector in $\mathbb{R}^n$, then

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\vec{v}\|^2.$$

Properties of the dot product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and k is a scalar, then:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ "symmetry" (dot product commutes)
2. $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} = 0$ ← zero scalar (zero vector)
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ } dot product distributes over addition
4. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
5. $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v})$ "homogeneity"
6. $\vec{v} \cdot \vec{v} \geq 0$, and $\vec{v} \cdot \vec{v} = 0$ if and only if $\vec{v} = \vec{0}$. "positivity"

Example 6. Use properties 1 and 3 above to prove property 4.

$$\begin{aligned}(\vec{u} + \vec{v}) \cdot \vec{w} &= \vec{w} \cdot (\vec{u} + \vec{v}) && \text{by property 1}\\ &= \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v} && \text{by property 3}\\ &= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} && \text{by property 1.}\end{aligned}$$

Example 7. Expand and simplify the vector expression.

$$\begin{aligned}(2\vec{u} + 3\vec{v}) \cdot (3\vec{u} - \vec{v}) &= 2\vec{u} \cdot (3\vec{u} - \vec{v}) + 3\vec{v} \cdot (3\vec{u} - \vec{v})\\ &= 6(\vec{u} \cdot \vec{u}) - 2(\vec{u} \cdot \vec{v}) + 9(\vec{v} \cdot \vec{u}) - 3(\vec{v} \cdot \vec{v})\\ &= 6\|\vec{u}\|^2 - 2(\vec{u} \cdot \vec{v}) + 9(\vec{u} \cdot \vec{v}) - 3\|\vec{v}\|^2\\ &= \underline{6\|\vec{u}\|^2 + 7(\vec{u} \cdot \vec{v}) - 3\|\vec{v}\|^2}.\end{aligned}$$

There are two important inequalities involving norms and distances in $\mathbb{R}^n$.

Cauchy-Schwarz Inequality. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

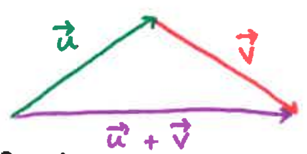
$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|.$$

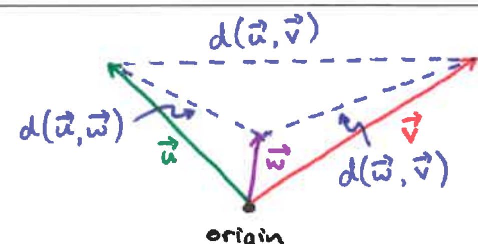
note: this implies that $-1 \leq \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} \leq 1$, so we can define the angle between $\vec{u}$ and $\vec{v}$ as $\theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}\right)$.

Triangle Inequality. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, then:

(a) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ triangle inequality for vectors

(b) $d(\vec{u}, \vec{v}) \leq d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v})$ triangle inequality for distances





Proof of (a).

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \quad \leftarrow \text{because } \|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$$

$$= (\vec{u} \cdot \vec{u}) + 2(\vec{u} \cdot \vec{v}) + (\vec{v} \cdot \vec{v}) \quad \text{apply absolute value to } \vec{u} \cdot \vec{v}$$

$$\leq \|\vec{u}\|^2 + 2|\vec{u} \cdot \vec{v}| + \|\vec{v}\|^2$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\| \, \|\vec{v}\| + \|\vec{v}\|^2$$

$$= (\|\vec{u}\| + \|\vec{v}\|)^2.$$

Because $\|\vec{u} + \vec{v}\| \geq 0$ and $\|\vec{u}\| + \|\vec{v}\| \geq 0$, we have $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

Example 8. Suppose that $\|\vec{u}\| = 4$ and $\|\vec{v}\| = 3$. What are the smallest and largest possible values of $\|\vec{u} + \vec{v}\|$?

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| = 4 + 3 = 7.$$

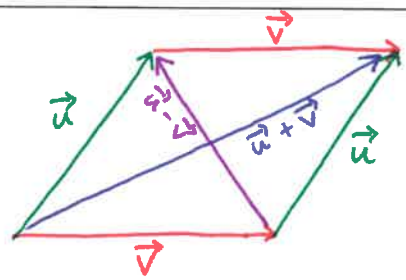
$$\|\vec{u}\| = \|(\vec{u} + \vec{v}) - \vec{v}\| \leq \|\vec{u} + \vec{v}\| + \|\vec{v}\|, \text{ so } 4 \leq \|\vec{u} + \vec{v}\| + 3.$$

Thus $\|\vec{u} + \vec{v}\| \geq 1$, and therefore $1 \leq \|\vec{u} + \vec{v}\| \leq 7$.

In plane geometry (that is, in $\mathbb{R}^2$), the sum of the squares of the two diagonals of a parallelogram equals the sum of the squares of the four sides. This result is also true more generally in $\mathbb{R}^n$.

Parallelogram equation for vectors. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).$$



Proof.

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\
&= (\vec{u}\cdot\vec{u}) + 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v}) + (\vec{u}\cdot\vec{u}) - 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v}) \\
&= 2(\vec{u}\cdot\vec{u}) + 2(\vec{v}\cdot\vec{v}) \\
&= 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).
\end{aligned}$$

Taking the difference of the squares of the two diagonals of a parallelogram instead gives a different expression for the dot product of two vectors.

Theorem. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2.$$

Proof.
$$\begin{aligned}
\tfrac{1}{4}\|\vec{u}+\vec{v}\|^2 - \tfrac{1}{4}\|\vec{u}-\vec{v}\|^2 &= \tfrac{1}{4}(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) - \tfrac{1}{4}(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\
&= \tfrac{1}{4}\left((\vec{u}\cdot\vec{u}) + 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v})\right) - \tfrac{1}{4}\left((\vec{u}\cdot\vec{u}) - 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v})\right) \\
&= \tfrac{1}{4}\left(4(\vec{u}\cdot\vec{v})\right) \\
&= \vec{u}\cdot\vec{v}.
\end{aligned}$$

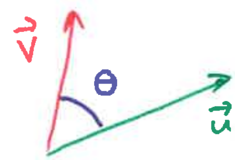
Section 3.3: Orthogonality

Objectives.

- Introduce the definition of orthogonality in $\mathbb{R}^n$.
- Represent lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$ using vector equations.
- Project a vector onto a line.
- Write a vector as the sum of two orthogonal components.

In Section 3.2, we defined the angle θ between two vectors $\vec{u}$ and $\vec{v}$ as

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$



The vectors $\vec{u}$ and $\vec{v}$ are orthogonal (or perpendicular) if

$$\vec{u}\cdot\vec{v} = 0.$$

note:
$\vec{u}\cdot\vec{v} > 0 \implies \theta$ is acute
$\vec{u}\cdot\vec{v} = 0 \implies \theta$ is a right angle
$\vec{u}\cdot\vec{v} < 0 \implies \theta$ is obtuse

Example 1. Show that the vectors $\vec{u} = (1, -2, 2, 5)$ and $\vec{v} = (3, 2, 3, -1)$ in $\mathbb{R}^4$ are orthogonal.

$$\vec{u}\cdot\vec{v} = (1,-2,2,5)\cdot(3,2,3,-1) = 3-4+6-5 = 0.$$

Thus $\vec{u}$ and $\vec{v}$ are orthogonal.

Notice that in $\mathbb{R}^n$, the standard basis vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are all orthogonal.

eg. $\vec{e}_1 \cdot \vec{e}_n = (1,0,\cdots,0)\cdot(0,0,\ldots,1) = 0.$

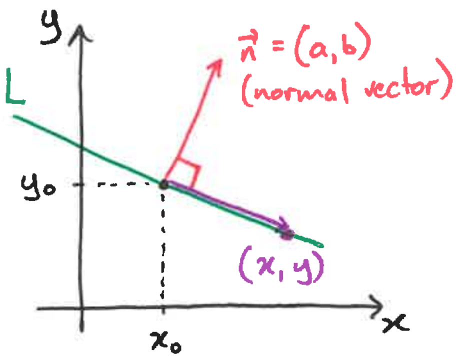
Pythagorean Theorem in $\mathbb{R}^n$. If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ then

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Proof. $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = (\vec{u}\cdot\vec{u}) + 2\underbrace{(\vec{u}\cdot\vec{v})}_{=0} + (\vec{v}\cdot\vec{v})$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2.$

A straight line in $\mathbb{R}^2$ can be described by specifying a point and a <u>normal</u> direction (that is, a vector orthogonal to the line).



If (x,y) is <u>any</u> point on the line L, then $(x-x_0, y-y_0)$ is orthogonal to $\vec{n}$.

$$\vec{n} \cdot (x-x_0, y-y_0) = 0$$

$$(a,b) \cdot (x-x_0, y-y_0) = 0$$

$$\boxed{a(x-x_0) + b(y-y_0) = 0}$$

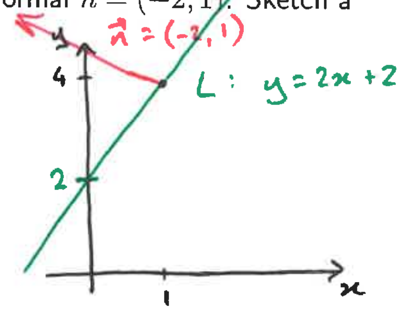
<u>or</u>: $ax + by + c = 0$.

Example 2. Write an equation for the line in $\mathbb{R}^2$ through the point $(1,4)$ with normal $\vec{n} = (-2,1)$. Sketch a diagram indicating the point, the normal vector, and the line.

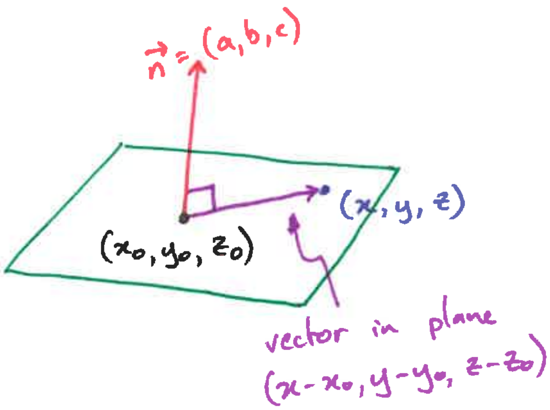
$$\vec{n} \cdot (x-x_0, y-y_0) = 0 \Rightarrow (-2,1) \cdot (x-1, y-4) = 0$$

$$\Rightarrow -2(x-1) + 1(y-4) = 0$$

$$\Rightarrow \underline{-2x + y - 2 = 0}.$$



The same idea can be used to write equations for planes in $\mathbb{R}^3$.



$$\vec{n} \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$(a,b,c) \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

<u>or</u>: $ax + by + cz + d = 0$

Example 3. Write an equation for the plane in $\mathbb{R}^3$ through the point $(2,-5,0)$ with normal $\vec{n} = (1,3,-1)$.

$$\vec{n} \cdot (x-x_0, y-y_0, z-z_0) = 0 \Rightarrow (1,3,-1) \cdot (x-2, y+5, z) = 0$$

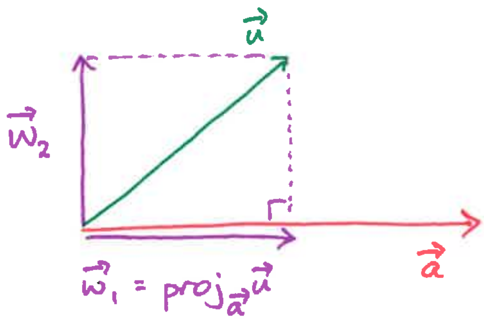
$$\Rightarrow (x-2) + 3(y+5) - z = 0$$

$$\Rightarrow \underline{x + 3y - z + 13 = 0}.$$

In Chapter 1, we introduced (orthogonal) projections onto the coordinate axes as examples or linear transformations. We can now extend this idea to (orthogonal) projections onto any line in $\mathbb{R}^n$.

Projection Theorem. If $\vec{u}$ and $\vec{a}$ are vectors in $\mathbb{R}^n$ with $\vec{a} \neq \vec{0}$, then $\vec{u}$ can be written in exactly one way as $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\vec{w}_1$ is parallel to $\vec{a}$ and $\vec{w}_2$ is orthogonal to $\vec{a}$. Specifically:

$$\vec{w}_1 = \mathrm{proj}_{\vec{a}}\vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\,\vec{a} \qquad \text{and} \qquad \vec{w}_2 = \vec{u} - \mathrm{proj}_{\vec{a}}\vec{u} = \vec{u} - \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\,\vec{a}.$$



why? $\vec{w}_1 = k\vec{a}$ and $\vec{w}_2 \cdot \vec{a} = 0$, so

$$\vec{u}\cdot\vec{a} = (\vec{w}_1 + \vec{w}_2)\cdot\vec{a} = \vec{w}_1\cdot\vec{a} + \vec{w}_2\cdot\vec{a}$$
$$= k\vec{a}\cdot\vec{a} + 0 = k\|\vec{a}\|^2.$$
$$\Longrightarrow k = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}, \text{ so}$$
$$\vec{w}_1 = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}, \quad \vec{w}_2 = \vec{u} - \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}.$$

Example 4. Let $\vec{u} = (1, 2, 3)$ and $\vec{a} = (4, -1, -1)$. Find the component of $\vec{u}$ parallel to $\vec{a}$ and the component of $\vec{u}$ orthogonal to $\vec{a}$.

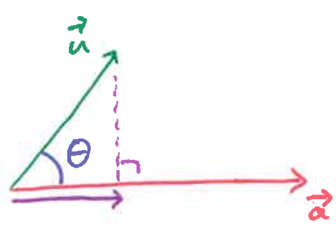
component $\parallel$ to $\vec{a}$:

$$\mathrm{proj}_{\vec{a}}\vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a} = \frac{(1,2,3)\cdot(4,-1,-1)}{(4,-1,-1)\cdot(4,-1,-1)}(4,-1,-1) = \frac{-1}{18}(4,-1,-1) = \left(-\frac{2}{9}, \frac{1}{18}, \frac{1}{18}\right).$$

component $\perp$ to $\vec{a}$:

$$\vec{u} - \mathrm{proj}_{\vec{a}}\vec{u} = (1,2,3) - \left(-\frac{2}{9}, \frac{1}{18}, \frac{1}{18}\right) = \left(\frac{11}{9}, \frac{35}{18}, \frac{53}{18}\right).$$

The norm of the orthogonal projection (of $\vec{u}$ onto $\vec{a}$ can be written either in terms of the two vectors or in terms of $\vec{u}$ and the angle θ between $\vec{u}$ and $\vec{a}$.



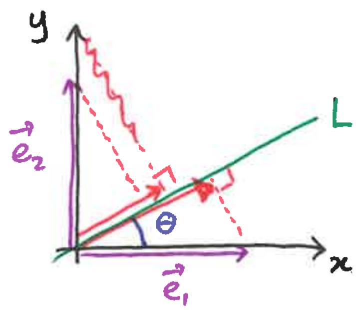
$\|\mathrm{proj}_{\vec{a}}\vec{u}\| = \|\vec{u}\|\cos\theta$

$$\|\mathrm{proj}_{\vec{a}}\vec{u}\| = \left\|\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}\right\| = \frac{|\vec{u}\cdot\vec{a}|}{\|\vec{a}\|^2}\|\vec{a}\| = \frac{|\vec{u}\cdot\vec{a}|}{\|\vec{a}\|}$$
$$= \frac{\|\vec{u}\|\|\vec{a}\|\cos\theta}{\|\vec{a}\|} = \|\vec{u}\|\cos\theta.$$

assuming θ is acute!!!

Example 5. Let L be a line through the origin in $\mathbb{R}^2$ that makes an angle θ with the positive x-axis.

(a) Find the projections of $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$ onto L.



$\vec{a} = (\cos\theta, \sin\theta)$ is a vector in the direction of L.

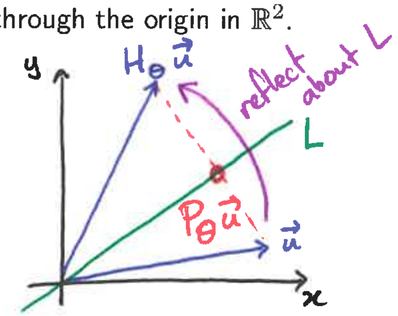
$$\operatorname{proj}_{\vec{a}} \vec{e}_1 = \frac{(1,0)\cdot(\cos\theta,\sin\theta)}{1^2}(\cos\theta,\sin\theta) = (\cos^2\theta, \cos\theta\sin\theta)$$

$$\operatorname{proj}_{\vec{a}} \vec{e}_2 = \frac{(0,1)\cdot(\cos\theta,\sin\theta)}{1^2}(\cos\theta,\sin\theta) = (\cos\theta\sin\theta, \sin^2\theta).$$

(b) Find the standard matrix P_θ for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that projects each point onto L.

$$P_\theta = \left[\operatorname{proj}_{\vec{a}} \vec{e}_1 \;\middle|\; \operatorname{proj}_{\vec{a}} \vec{e}_2\right] = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}.$$

We can use the previous example to find a linear transformation that reflects a vector/point about a line through the origin in $\mathbb{R}^2$.



$$P_\theta \vec{u} = \frac{1}{2}(H_\theta \vec{u} + \vec{u})$$

$$\Rightarrow \quad P_\theta = \frac{1}{2}(H_\theta + I)$$

$$\Rightarrow \quad H_\theta = 2P_\theta - I = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Example 6. Let $\vec{x} = (4,1)$ and let L be the line through the origin that makes an angle of $\pi/3$ with the positive x-axis.

(a) Find the projection of $\vec{x}$ onto L.

$$P_{\frac{\pi}{3}} = \begin{bmatrix} \cos^2\frac{\pi}{3} & \cos\frac{\pi}{3}\sin\frac{\pi}{3} \\ \cos\frac{\pi}{3}\sin\frac{\pi}{3} & \sin^2\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}, \quad \text{so} \quad P_{\frac{\pi}{3}}(4,1) = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\sqrt{3}}{4} \\ \sqrt{3} + \frac{3}{4} \end{bmatrix}$$

(b) Find the reflection of $\vec{x}$ about L.

$$H_{\frac{\pi}{3}} = \begin{bmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \text{so} \quad H_{\frac{\pi}{3}}(4,1) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + \frac{\sqrt{3}}{2} \\ 2\sqrt{3} + \frac{1}{2} \end{bmatrix}$$

Distance problems.

The distance between a point and a line in $\mathbb{R}^2$ or between a point and a plane in $\mathbb{R}^3$ can be found using projections.

Theorem.

1. In $\mathbb{R}^2$, the distance between the point $P_0 = (x_0, y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

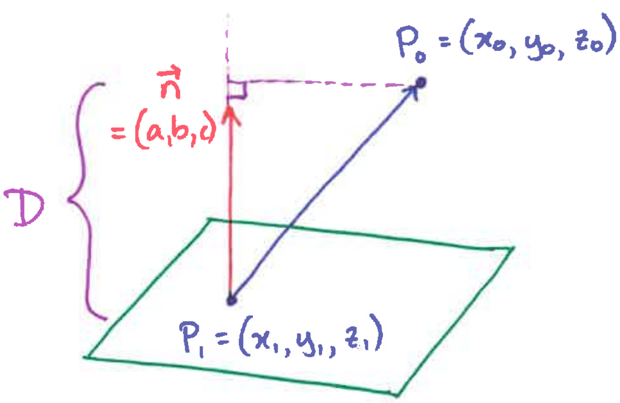
normal is $\vec{n} = (a,b,c)$

2. In $\mathbb{R}^3$, the distance between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Proof of 2.

Choose $P_1 = (x_1, y_1, z_1)$ in the plane, and project $\overrightarrow{P_1P_0}$ onto $\vec{n}$.



b/c P_1 is in the plane,

$ax_1 + by_1 + cz_1 + d = 0.$

$\Rightarrow -ax_1 - by_1 - cz_1 = d.$

$$D = \|\text{proj}_{\vec{n}} \overrightarrow{P_1P_0}\|$$

$$= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a,b,c)|}{\|(a,b,c)\|}$$

$$= \frac{|ax_0 - ax_1 + by_0 - by_1 + cz_0 - cz_1|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 7. Find the distance in $\mathbb{R}^2$ between the point $(1, -1)$ and the line $x + 2y = 3$.

$a = 1, \ b = 2, \ c = -3$

$$D = \frac{|1(1) + 2(-1) + (-3)|}{\sqrt{1^2 + 2^2}} = \frac{|-4|}{\sqrt{5}} = \frac{4}{\sqrt{5}}.$$

Section 3.4: The Geometry of Linear Systems

Objectives.

- Write vector and parametric equations for lines and planes in $\mathbb{R}^n$.
- Express a line segment in vector form.

In Section 3.3, we saw how the dot product allows us to write vector and scalar equations for a line in $\mathbb{R}^2$ or a plane in $\mathbb{R}^3$. Specifically:

- the line in $\mathbb{R}^2$ through the point $\vec{x}_0 = (x_0, y_0)$ and normal to the vector $\vec{n} = (a, b)$ is

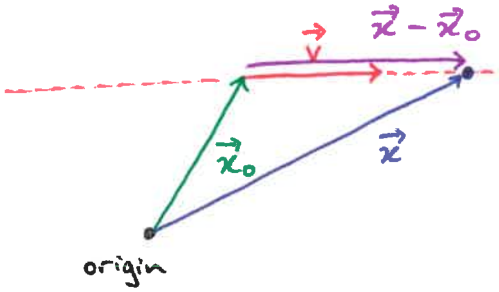
$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \quad \text{or} \quad a(x - x_0) + b(y - y_0) = 0.$$

- the plane in $\mathbb{R}^3$ through the point $\vec{x}_0 = (x_0, y_0, z_0)$ and normal to the vector $\vec{n} = (a, b, c)$ is

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \quad \text{or} \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

In this section, we will explore how the equation of a line in higher dimensions can be written using a point on the line and a direction parallel to the line, and how the equation of a plane in higher dimensions can be written using a point on the plane and two (non-parallel!) directions parallel to the plane.

Suppose that $\vec{x}$ is a general point on the line through the point $\vec{x}_0$ and parallel to the vector $\vec{v}$.



A vector on this line is a scalar multiple of $\vec{v}$.

$$\vec{x} - \vec{x}_0 = t\,\vec{v}$$

parameter

$$\Rightarrow \quad \vec{x} = \vec{x}_0 + t\,\vec{v}$$

gen. pt. = fixed pt. + parameter · direction

Example 1. Let L be the line in $\mathbb{R}^3$ through the point $\vec{x}_0 = (3, -1, 5)$ and parallel to the vector $\vec{v} = (-2, 1, 2)$.

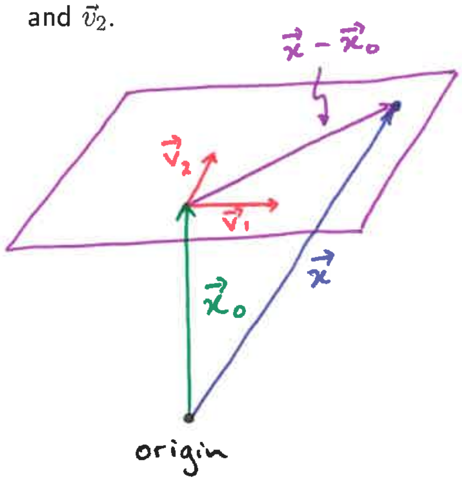
(a) Find a vector equation for the line L.

$$\vec{x} = \vec{x}_0 + t\,\vec{v} = (3, -1, 5) + t\,(-2, 1, 2). = (3 - 2t, -1 + t, 5 + 2t).$$

(b) Find parametric equations for the line L.

$$x = 3 - 2t, \quad y = -1 + t, \quad z = 5 + 2t$$

Suppose $\vec{x}$ is a general point on the plane through the point $\vec{x}_0$ and parallel to the (non-parallel) vectors $\vec{v}_1$ and $\vec{v}_2$.



A vector $\vec{x} - \vec{x}_0$ in the plane is a linear combination of $\vec{v}_1$ and $\vec{v}_2$

$$\vec{x} - \vec{x}_0 = t_1 \vec{v}_1 + t_2 \vec{v}_2$$

$$\Rightarrow \vec{x} = \vec{x}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2 .$$

Example 2. Consider the point $\vec{x}_0 = (1, 4, 0, -3)$ in $\mathbb{R}^4$ and the vectors $\vec{v}_1 = (2, -1, 1, 0)$ and $\vec{v}_2 = (3, -6, 5, 2)$.

(a) Find a vector equation for the plane through $\vec{x}_0$ and parallel to both $\vec{v}_1$ and $\vec{v}_2$.

$$\vec{x} = \vec{x}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2 = (1,4,0,-3) + t_1(2,-1,1,0) + t_2(3,-6,5,2).$$

(b) Find parametric equations for the plane in part (a).

$$w = 1 + 2t_1 + 3t_2, \quad x = 4 - t_1 - 6t_2, \quad y = t_1 + 5t_2, \quad z = -3 + 2t_2.$$

Example 3. The scalar equation $x + 2y + 3z = 4$ represents a plane in $\mathbb{R}^3$.

(a) Find parametric equations for the plane. $\longrightarrow$ use two variables as parameters!!!

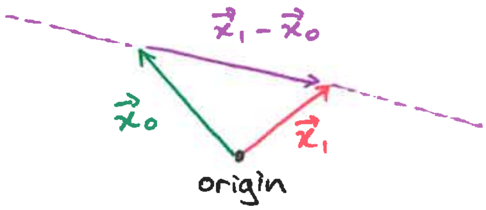
Let $y = t_1$ and $z = t_2$. Then

$$x = 4 - 2t_1 - 3t_2.$$

(b) Find a vector equation for the plane.

$$\vec{x} = (4 - 2t_1 - 3t_2, t_1, t_2) = (4,0,0) + t_1(-2,1,0) + t_2(-3,0,1).$$

Any two distinct points $\vec{x}_0$ and $\vec{x}_1$ in $\mathbb{R}^n$ determine a unique line:



$$\vec{x} = \vec{x}_0 + t\underline{(\vec{x}_1 - \vec{x}_0)}$$

i.e. $\vec{v} = \vec{x}_1 - \vec{x}_0$ is a direction parallel to the line.

or

$$\vec{x} = (1-t)\,\vec{x}_0 + t\,\vec{x}_1$$

Example 4. Consider the two points $\vec{x}_0 = (1, -1)$ and $\vec{x}_1 = (0, 3)$ in $\mathbb{R}^2$.

(a) Find a vector equation for the line through $\vec{x}_0$ and $\vec{x}_1$.

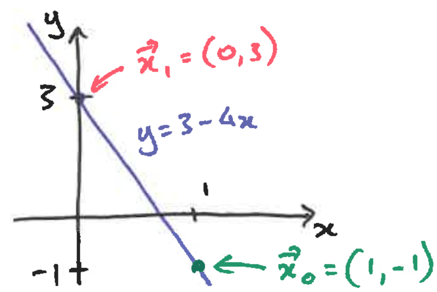
$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) = (1,-1) + t\big(0-1,\ 3-(-1)\big) = (1,-1) + t(-1,4).$$

(b) Write a scalar equation for the line in part (a).

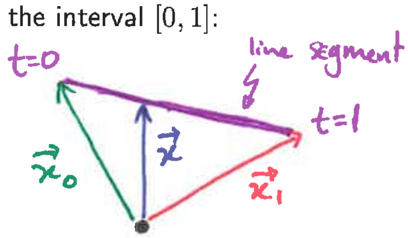
From $x = 1-t$ and $y = -1+4t$, we have

$t = 1-x$ and $4t = 1+y$.

Thus $4(1-x) = 1+y$, or $\underline{y = 3-4x}$.



To describe the line segment connecting two points $\vec{x}_0$ and $\vec{x}_1$ in $\mathbb{R}^n$, we can restrict the values of the parameter t to the interval $[0, 1]$:



$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0), \quad 0 \le t \le 1$$

or

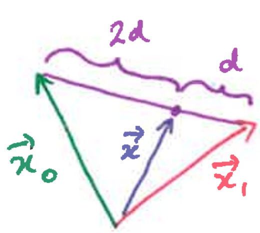
$$\vec{x} = (1-t)\vec{x}_0 + t\,\vec{x}_1, \quad 0 \le t \le 1$$

Example 5. Consider the two points $\vec{x}_0 = (1, -4, -2, 5)$ and $\vec{x}_1 = (4, -2, 7, 2)$.

(a) Find an equation for the line segment from $\vec{x}_0$ to $\vec{x}_1$.

$$\vec{x} = (1-t)(1,-4,-2,5) + t(4,-2,7,2), \quad 0 \le t \le 1.$$

(b) Find the point on this line segment for which the distance to $\vec{x}_0$ is twice the distance to $\vec{x}_1$.



• use $t = \frac{2}{3}$ (i.e. $\frac{2}{3}$ of distance from $\vec{x}_0$ to $\vec{x}_1$)

$$\vec{x} = \left(1-\tfrac{2}{3}\right)(1,-4,-2,5) + \tfrac{2}{3}(4,-2,7,2)$$

$$= \left(\tfrac{1}{3}, -\tfrac{4}{3}, -\tfrac{2}{3}, \tfrac{5}{3}\right) + \left(\tfrac{8}{3}, -\tfrac{4}{3}, \tfrac{14}{3}, \tfrac{4}{3}\right)$$

$$= \left(3, -\tfrac{8}{3}, 4, 3\right).$$

Recall that a homogeneous linear equation has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

or: $\vec{a} \cdot \vec{x} = 0$, where $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$.

Notice from this that every vector that satisfies a homogeneous linear equation is orthogonal to the coefficient vector. In particular, any solution to the matrix equation $A\vec{x} = \vec{0}$ is orthogonal to every row of the matrix A.

Theorem. If A is an $m \times n$ matrix, then the set of solutions to the homogeneous linear system $A\vec{x} = \vec{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row of A.

Example 6. The linear system

$$\begin{bmatrix} 1 & 5 & -10 & 0 & 2 \\ 3 & -2 & 0 & 2 & 1 \\ 4 & 2 & 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solution $x_1 = -2t$, $x_2 = 2s$, $x_3 = s + t$, $x_4 = 2s$, $x_5 = 6t$. Show that the vector

$$\vec{x} = (-2t, 2s, s+t, 2s, 6t) \longleftarrow \text{solutions for system!!!}$$

is orthogonal to every row of the coefficient matrix for the system.

$$\begin{aligned} \vec{r}_1 \cdot \vec{x} &= (1, 5, -10, 0, 2) \cdot (-2t, 2s, s+t, 2s, 6t) \\ &= -2t + 10s - 10(s+t) + 0(2s) + 2(6t) \\ &= -2t + 10s - 10s - 10t + 12t = \underline{0.} \end{aligned}$$

$$\begin{aligned} \vec{r}_2 \cdot \vec{x} &= (3, -2, 0, 2, 1) \cdot (-2t, 2s, s+t, 2s, 6t) \\ &= 3(-2t) - 2(2s) + 0(s+t) + 2(2s) + 1(6t) \\ &= -6t - 4s + 4s + 6t = \underline{0.} \end{aligned}$$

$$\begin{aligned} \vec{r}_3 \cdot \vec{x} &= (4, 2, 2, -3, 1) \cdot (-2t, 2s, s+t, 2s, 6t) \\ &= -8t + 4s + 2s + 2t - 6s + 6t = \underline{0.} \end{aligned}$$

Section 3.5: Cross Product

Objectives.

- Introduce the cross product of two vectors in $\mathbb{R}^3$.
- Interpret the cross product geometrically.
- Study some properties of the cross product.

The cross product of two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is

note: vector x vector = vector.

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix},\ -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix},\ \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

(Note that the cross product is only defined for vectors in $\mathbb{R}^3$.)

Example 1. Compute $\vec{u} \times \vec{v}$ for the vectors $\vec{u} = (2, 3, -2)$ and $\vec{v} = (1, 4, 1)$.

$$\vec{u} \times \vec{v} = ((3)(1) - (-2)(4),\ (-2)(1) - (2)(1),\ (2)(4) - (3)(1))$$

$$= (11, -4, 5).$$

The cross product can also be expressed as a 3×3 determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

Example 2. Compute $\vec{v} \times \vec{u}$ for the vectors in Example 1. What do you notice?

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

$$\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 1 \\ 2 & 3 & -2 \end{vmatrix} = -8\vec{i} + 2\vec{j} + 3\vec{k} - 8\vec{k} - 3\vec{i} + 2\vec{j}$$

$$= -11\vec{i} + 4\vec{j} - 5\vec{k}$$

$$= (-11, 4, -5).$$

$$\left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 4 & 1 & 1 & 4 \\ 2 & 3 & -2 & 2 & 3 \end{array} \right.$$

Properties of the cross product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$ and k is a scalar, then:

1. $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$ anticommutative
2. $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v}) + (\vec{u}\times\vec{w})$ $\Big\}$ cross products distributes over addition
3. $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$
4. $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})$ scalar multiples behave "nicely"
5. $\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = \vec{0}$
6. $\vec{u}\times\vec{u} = \vec{0}$

Proof of 1. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Then

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = -(\vec{v}\times\vec{u}).$$

Example 3. Show that $(\vec{u}+k\vec{v})\times\vec{v} = \vec{u}\times\vec{v}$.

$$(\vec{u}+k\vec{v})\times\vec{v} = (\vec{u}\times\vec{v}) + (k\vec{v}\times\vec{v}) = (\vec{u}\times\vec{v}) + k(\cancelto{\vec{0}}{\vec{v}\times\vec{v}})$$

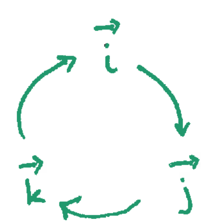
$$= (\vec{u}\times\vec{v}) + k\vec{0} = \vec{u}\times\vec{v}.$$

Example 4. Compute the following cross products, where $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, and $\vec{k} = (0,0,1)$.

(a) $\vec{i}\times\vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$

(b) $\vec{j}\times\vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}$

(c) $\vec{k}\times\vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 0 & 0 \\ 1 & 0 \end{matrix} = \vec{j}$



$\vec{i}\times\vec{i} = \vec{0}$

$\vec{i}\times\vec{j} = \vec{k}$

$\vec{i}\times\vec{k} = -\vec{j}$

An important property of the cross product is that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Relationships between the dot product and the cross product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$, then:

1. $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ i.e. $\vec{u}$ is orthogonal to $\vec{u} \times \vec{v}$
2. $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$ Lagrange's identity
3. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ scalar triple product
4. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$ vector triple product

Proof of 1. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Then:

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = (u_1, u_2, u_3) \cdot (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$$

$$= \underline{u_1 u_2 v_3} - \underline{u_1 u_3 v_2} + \underline{u_2 u_3 v_1} - \underline{u_2 u_1 v_3} + \underline{u_3 u_1 v_2} - \underline{u_3 u_2 v_1}$$

$$= 0.$$

Therefore $\vec{u}$ and $\vec{u} \times \vec{v}$ are orthogonal.

Example 5. For the vectors $\vec{u} = (2, 3, -2)$ and $\vec{v} = (1, 4, 1)$ in Example 1, confirm that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

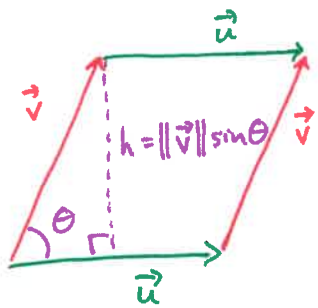
<u>recall</u>: $\vec{u} \times \vec{v} = (11, -4, 5)$.

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = (2, 3, -2) \cdot (11, -4, 5) = 22 - 12 - 10 = 0$$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = (1, 4, 1) \cdot (11, -4, 5) = 11 - 16 + 5 = 0$$

Thus both $\vec{u}$ and $\vec{v}$ are orthogonal to $\vec{u} \times \vec{v}$.

The norm of $\vec{u} \times \vec{v}$ is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.



(from Lagrange:) $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

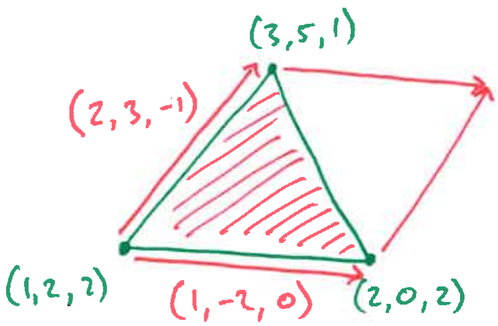
$$= \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2\theta$$

$$= \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2\theta)$$

$$= \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2\theta$$

Therefore: $\boxed{\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta}$ ← area of parallelogram.

Example 6. Find the area of the triangle with vertices $(1, 2, 2)$, $(3, 5, 1)$, and $(2, 0, 2)$.

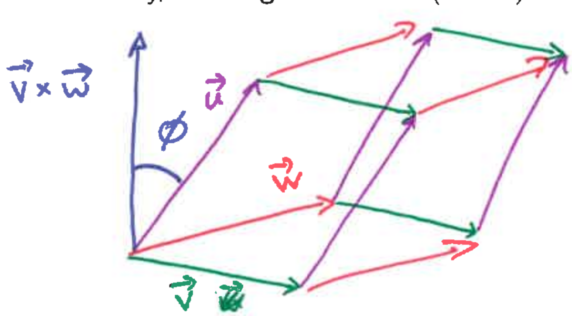


$$\text{area} = \frac{1}{2} \| (1,-2,0) \times (2,3,-1) \|$$

$$= \frac{1}{2} \| (2,1,7) \|$$

$$= \underline{\frac{1}{2}\sqrt{54}}.$$

Similarly, the magnitude of $\vec{u} \cdot (\vec{v} \times \vec{w})$ is the volume of the parallelipiped spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$.



$$\text{volume} = \text{height} \times \text{area of base}$$

$$= \left(\|\vec{u}\| |\cos\phi|\right) \left(\|\vec{v} \times \vec{w}\|\right)$$

$$= \|\vec{u}\| \|\vec{v} \times \vec{w}\| |\cos\phi|$$

$$= |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

Example 7. Find the volume of the parallelipiped spanned by $(1, 2, 2)$, $(3, 5, 1)$, and $(2, 0, 2)$.

$$\text{volume} = |(1,2,2) \cdot ((3,5,1) \times (2,0,2))| = |(1,2,2) \cdot (10,-4,-10)|$$

$$= |10 - 8 - 20| = |-18| = \underline{18}.$$

Theorem. The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ lie in the same plane if and only if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

i.e. the volume spanned by $\vec{u}, \vec{v}, \vec{w}$ is zero, so these vectors determine a flat surface rather than a parallelepiped.

Section 3.3: Orthogonal projections in $\mathbb{R}^3$

The orthogonal projections of a vector $\vec{x} = (x, y, z)$ in $\mathbb{R}^3$ onto each of the coordinate axes are given by:

$$\begin{aligned} T_x(\vec{x}) &= (x,0,0) && \textit{projection onto } x\textit{-axis,} \\ T_y(\vec{x}) &= (0,y,0) && \textit{projection onto } y\textit{-axis,} \\ T_z(\vec{x}) &= (0,0,z) && \textit{projection onto } z\textit{-axis.} \end{aligned}$$

Problem 1. Let $\vec{x} = (x, y, z)$ be a vector in $\mathbb{R}^3$.

(a) Show that the vectors $T_x(\vec{x})$ and $T_y(\vec{x})$ are orthogonal.

$$\begin{aligned} T_x(\vec{x}) \cdot T_y(\vec{x}) &= (x,0,0)\cdot(0,y,0) \\ &= x\cdot 0 + 0\cdot y + 0\cdot 0 \\ &= 0. \end{aligned}$$

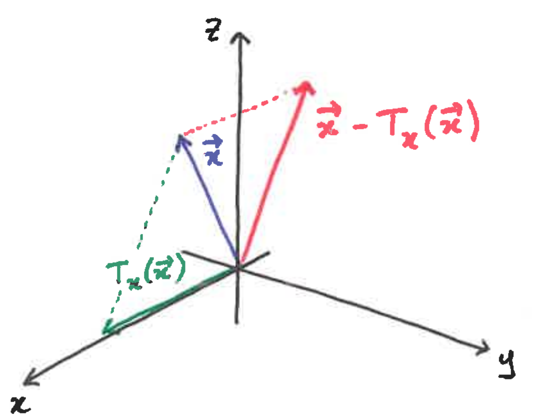
Thus $T_x(\vec{x})$ and $T_y(\vec{x})$ are orthogonal.

(b) Show that the vectors $T_x(\vec{x})$ and $\vec{x} - T_x(\vec{x})$ are orthogonal.

$$\begin{aligned} T_x(\vec{x})\cdot(\vec{x} - T_x(\vec{x})) &= (x,0,0)\cdot((x,y,z)-(x,0,0)) \\ &= (x,0,0)\cdot(0,y,z) \\ &= 0. \end{aligned}$$

Thus $T_x(\vec{x})$ and $\vec{x} - T_x(\vec{x})$ are orthogonal.

(c) Sketch a diagram showing $\vec{x}$, $T_x(\vec{x})$, and $\vec{x} - T_x(\vec{x})$.



Section 3.4: Transformations of lines in $\mathbb{R}^n$

Recall that a line in $\mathbb{R}^n$ can be represented by the equation

$$\vec{x} = \vec{x}_0 + t\vec{v},$$

where $\vec{x}$ is a general point on the line, $\vec{x}_0$ is a fixed point on the line, and $\vec{v}$ is a nonzero vector parallel to the line.

Problem 2. Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator, so that A is an invertible $n \times n$ matrix.

(a) Show that the image of the line $\vec{x} = \vec{x}_0 + t\vec{v}$ in $\mathbb{R}^n$ under the transformation T_A is also a line in $\mathbb{R}^n$.

$$\begin{aligned} T_A(\vec{x}) &= T_A(\vec{x}_0 + t\vec{v}) \\ &= T_A(\vec{x}_0) + t\,T_A(\vec{v}) \\ &= A\vec{x}_0 + t A\vec{v}. \end{aligned}$$

Because $A\vec{x}_0$ is a vector in $\mathbb{R}^n$, and $A\vec{v}$ is a nonzero vector in $\mathbb{R}^n$ (since A is invertible), this represents a line in $\mathbb{R}^n$.

(b) Let $A = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$. Find vector and parametric equations for the image of the line $\vec{x} = (1,3) + t(2,-1)$ under multiplication by A.

$$A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}, \quad A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

The image of the line $\vec{x} = (1,3) + t(2,-1)$ is the line $\vec{x} = (5,-9) + t(3,10)$.

The parametric equations are $x = 5 + 3t$ and $y = -9 + 10t$.

Section 10.1: Constructing Curves and Surfaces Through Specified Points

Lines in $\mathbb{R}^2$

Any two distinct points (x_1, y_1), (x_2, y_2) in $\mathbb{R}^2$ lie a (unique) line $c_1x + c_2y + c_3 = 0$, where at least one of c_1 and c_2 is not zero. This implies that the homogeneous linear system

$$\begin{aligned} xc_1 + \ \ yc_2 + c_3 &= 0 \\ x_1c_1 + y_1c_2 + c_3 &= 0 \\ x_2c_1 + y_2c_2 + c_3 &= 0 \end{aligned}$$

has a non-trivial solution; equivalently the determinant of the coefficient matrix is zero, which gives the following equation for the line through (x_1, y_1) and (x_2, y_2).

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Problem 3. Consider the line in $\mathbb{R}^2$ through the two points $(3, 1)$ and $(5, -8)$.

(a) Use the determinant above to find an equation for the line.

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 5 & -8 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad x\begin{vmatrix} 1 & 1 \\ -8 & 1 \end{vmatrix} - y\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 5 & -8 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(1+8) - y(3-5) + (-24-5) = 0$$

$$\Rightarrow \quad \underline{9x + 2y - 29 = 0.}$$

(b) Find the points where the line intersects each of the coordinate axes.

If $y=0$ then $x = \frac{29}{9}$. If $x=0$ then $y = \frac{29}{2}$.

The line intersects the axes at $\left(\frac{29}{9}, 0\right)$ and $\left(0, \frac{29}{2}\right)$.

(c) Graph the equation from part (a) to confirm that the line passes through the two given points.

Circles in $\mathbb{R}^2$

The same method can be used to find a determinant equation for the unique circle

$$c_1(x^2+y^2)+c_2x+c_3y+c_4=0$$

through three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) not on the same line.

Problem 4. Suppose the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) all lie on the circle $c_1(x^2+y^2)+c_2x+c_3y+c_4=0$.

(a) Set up a homogeneous system of linear equations in c_1, c_2, c_3, and c_4 satisfied by the three given points and a general point (x, y) on the same circle.

$$\begin{aligned} c_1(x^2+y^2)+c_2x+c_3y+c_4&=0\\ c_1(x_1^2+y_1^2)+c_2x_1+c_3y_1+c_4&=0\\ c_1(x_2^2+y_2^2)+c_2x_2+c_3y_2+c_4&=0\\ c_1(x_3^2+y_3^2)+c_2x_3+c_3y_3+c_4&=0 \end{aligned}$$

(b) The system in part (a) has non-trivial solutions. Write a determinant equation to represent this.

$$\begin{vmatrix} x^2+y^2 & x & y & 1\\ x_1^2+y_1^2 & x_1 & y_1 & 1\\ x_2^2+y_2^2 & x_2 & y_2 & 1\\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

(c) Find the center and the radius of the circle passing through $(2, -2)$, $(3, 5)$, and $(-4, 6)$.

$$\begin{vmatrix} x^2+y^2 & x & y & 1\\ 8 & 2 & -2 & 1\\ 34 & 3 & 5 & 1\\ 52 & -4 & +6 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 50x^2 + 100x + 50y^2 - 200y - 1000 = 0$$

$$\Rightarrow \quad x^2+2x+y^2-4y = 20$$

$$\Rightarrow \quad (x+1)^2+(y-2)^2 = 25 \qquad \text{center} = (-1, 2), \ \text{radius} = 5.$$

(d) Graph the equation from part (c) to confirm that the circle passes through the three given points.

Conic sections in $\mathbb{R}^2$

A general conic section in $\mathbb{R}^2$ has equation

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0,$$

and is determined by five distinct points in the plane.

Problem 5. (a) Find a determinant equation for the conic section through the five distinct points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5).$$

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0$$

(b) Find an equation for the conic section through the points $(0,0)$, $(0,-1)$, $(2,0)$, $(2,-5)$, and $(4,-1)$.

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 4 & 0 & 0 & 2 & 0 & 1 \\ 4 & -10 & 25 & 2 & -5 & 1 \\ 16 & -4 & 1 & 4 & -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 160x^2 + 320xy + 320y^2 - 320x + 320y = 0$$

$$\Rightarrow \quad x^2 + 2xy + y^2 - 2x + 2y = 0$$

(c) Graph the equation from part (b). What type of conic section is this?

a parabola!!!

Planes in $\mathbb{R}^3$

A plane in $\mathbb{R}^3$ has the scalar equation $c_1x + c_2y + c_3z + c_4 = 0$, and is determined by three points not on the same line.

Problem 6. (a) Find a determinant equation for the plane through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

(b) Find a scalar equation of the plane through the points $(2, 1, 3)$, $(2, -1, -1)$, and $(1, 1, 2)$.

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 3 & 1 \\ 2 & -1 & -1 & 1 \\ 1 & 1 & 2 & 1 \end{vmatrix} = 0 \implies 2x + 4y - 2z - 1 = 0.$$

(c) Graph the equation from part (c) to confirm that the plane passes through the three given points.

Spheres in $\mathbb{R}^3$

A sphere in $\mathbb{R}^3$ has equation

$$c_1(x^2 + y^2 + z^2) + c_2x + c_3y + c_4z + c_5 = 0,$$

and is determined by four points not in the same plane.

Problem 7. (a) Find a determinant equation for the sphere through the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4).

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

(b) Find an equation of the sphere through the points $(0, 1, -2)$, $(1, 3, 1)$, $(2, -1, 0)$, and $(3, 1, -1)$.

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 5 & 0 & 1 & -2 & 1 \\ 11 & 1 & 3 & 1 & 1 \\ 5 & 2 & -1 & 0 & 1 \\ 11 & 3 & 1 & -1 & 1 \end{vmatrix} = 0 \implies \underline{x^2 - 2x + y^2 - 2y + z^2 = 3.}$$

(c) Graph the equation from part (c) to confirm that the sphere passes through the four given points.