Section 3.5: Cross Product

Objectives.

- Introduce the cross product of two vectors in $\mathbb{R}^3$.
- Interpret the cross product geometrically.
- Study some properties of the cross product.

The cross product of two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is

(Note that the cross product is only defined for vectors in $\mathbb{R}^3$.)

Example 1. Compute $\vec{u} \times \vec{v}$ for the vectors $\vec{u} = (2, 3, -2)$ and $\vec{v} = (1, 4, 1)$.

The cross product can also be expressed as a 3×3 determinant:

Example 2. Compute $\vec{v} \times \vec{u}$ for the vectors in Example 1. What do you notice?

Properties of the cross product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$ and k is a scalar, then:

1. $\vec{u} \times \vec{v} =$

2. $\vec{u} \times (\vec{v} + \vec{w}) =$

3. $(\vec{u} + \vec{v}) \times \vec{w} =$

4. $k(\vec{u} \times \vec{v}) =$

5. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} =$

6. $\vec{u} \times \vec{u} =$

Proof of 1.

Example 3. Show that $\vec{u} + k\vec{v}) \times \vec{v} = \vec{u} \times \vec{v}$.

Example 4. Compute the following cross products, where $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$.

(a) $\vec{i} \times \vec{j} =$

(b) $\vec{j} \times \vec{k} =$

(c) $\vec{k} \times \vec{i} =$

An important property of the cross product is that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Relationships between the dot product and the cross product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$, then:

1. $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$
2. $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$
3. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
4. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Proof of 1.

Example 5. For the vectors $\vec{u} = (2, 3, -2)$ and $\vec{v} = (1, 4, 1)$ in Example 1, confirm that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

The norm of $\vec{u} \times \vec{v}$ is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.

Example 6. Find the area of the triangle with vertices $(1, 2, 2)$, $(3, 5, 1)$, and $(2, 0, 2)$.

Similarly, the magnitude of $\vec{u} \cdot (\vec{v} \times \vec{w})$ is the volume of the parallelipiped spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Example 7. Find the volume of the parallelipiped spanned by $(1, 2, 2)$, $(3, 5, 1)$, and $(2, 0, 2)$.

<u>**Theorem.**</u> The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ lie in the same plane if and only if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.