Section 3.4: The Geometry of Linear Systems

Objectives.

- Write vector and parametric equations for lines and planes in $\mathbb{R}^n$.
- Express a line segment in vector form.

In Section 3.3, we saw how the dot product allows us to write vector and scalar equations for a line in $\mathbb{R}^2$ or a plane in $\mathbb{R}^3$. Specifically:

- the line in $\mathbb{R}^2$ through the point $\vec{x}_0 = (x_0, y_0)$ and normal to the vector $\vec{n} = (a, b)$ is

- the plane in $\mathbb{R}^3$ through the point $\vec{x}_0 = (x_0, y_0, z_0)$ and normal to the vector $\vec{n} = (a, b, c)$ is

In this section, we will explore how the equation of a line in higher dimensions can be written using a point on the line and a direction parallel to the line, and how the equation of a plane in higher dimensions can be written using a point on the plane and two (non-parallel!) directions parallel to the plane.

Suppose that $\vec{x}$ is a general point on the line through the point $\vec{x}_0$ and parallel to the vector $\vec{v}$.

Example 1. Let L be the line in $\mathbb{R}^3$ through the point $\vec{x}_0 = (3, -1, 5)$ and parallel to the vector $\vec{v} = (-2, 1, 2)$.

(a) Find a vector equation for the line L.

(b) Find parametric equations for the line L.

Suppose $\vec{x}$ is a general point on the plane through the point $\vec{x}_0$ and parallel to the (non-parallel) vectors $\vec{v}_1$ and $\vec{v}_2$.

Example 2. Consider the point $\vec{x}_0 = (1, 4, 0, -3)$ in $\mathbb{R}^4$ and the vectors $\vec{v}_1 = (2, -1, 1, 0)$ and $\vec{v}_2 = (3, -6, 5, 2)$.

(a) Find a vector equation for the plane through $\vec{x}_0$ and parallel to both $\vec{v}_1$ and $\vec{v}_2$.

(b) Find parametric equations for the plane in part (a).

Example 3. The scalar equation $x + 2y + 3z = 4$ represents a plane in $\mathbb{R}^3$.

(a) Find parametric equations for the plane.

(b) Find a vector equation for the plane.

Any two distinct points $\vec{x}_0$ and $\vec{x}_1$ in $\mathbb{R}^n$ determine a unique line:

Example 4. Consider the two points $\vec{x}_0 = (1, -1)$ and $\vec{x}_1 = (0, 3)$ in $\mathbb{R}^2$.

(a) Find a vector equation for the line through $\vec{x}_0$ and $\vec{x}_1$.

(b) Write a scalar equation for the line in part (a).

To describe the line segment connecting two points $\vec{x}_0$ and $\vec{x}_1$ in $\mathbb{R}^n$, we can restrict the values of the parameter t to the interval $[0, 1]$:

Example 5. Consider the two points $\vec{x}_0 = (1, -4, -2, 5)$ and $\vec{x}_1 = (4, -2, 7, 2)$.

(a) Find an equation for the line segment from $\vec{x}_0$ to $\vec{x}_1$.

(b) Find the point on this line segment for which the distance to $\vec{x}_0$ is twice the distance to $\vec{x}_1$.

Recall that a homogeneous linear equation has the form

Notice from this that every vector that satisfies a homogeneous linear equation is orthogonal to the coefficient vector. In particular, any solution to the matrix equation $A\vec{x} = \vec{0}$ is orthogonal to every row of the matrix A.

Theorem. If A is an $m \times n$ matrix, then the set of solutions to the homogeneous linear system $A\vec{x} = \vec{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row of A.

Example 6. The linear system

$$\begin{bmatrix} 1 & 5 & -10 & 0 & 2 \\ 3 & -2 & 0 & 2 & 1 \\ 4 & 2 & 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solution $x_1 = -2t$, $x_2 = 2s$, $x_3 = s + t$, $x_4 = 2s$, $x_5 = 6t$. Show that the vector

$$\vec{x} = (-2t, 2s, s + t, 2s, 6t)$$

is orthogonal to every row of the coefficient matrix for the system.