Section 3.3: Orthogonality

Objectives.

- Introduce the definition of orthogonality in $\mathbb{R}^n$.
- Represent lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^n$ using vector equations.
- Project a vector onto a line.
- Write a vector as the sum of two orthogonal components.

In Section 3.2, we defined the angle θ between two vectors $\vec{u}$ and $\vec{v}$ as

The vectors $\vec{u}$ and $\vec{v}$ are orthogonal (or perpendicular) if

Example 1. Show that the vectors $\vec{u} = (1, -2, 2, 5)$ and $\vec{v} = (3, 2, 3, -1)$ in $\mathbb{R}^4$ are orthogonal.

Notice that in $\mathbb{R}^n$, the standard basis vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ are all orthogonal.

Pythagorean Theorem in $\mathbb{R}^n$. If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ then

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Proof.

A straight line in $\mathbb{R}^2$ can be described by specifying a point and a normal direction (that is, a vector orthogonal to the line).

Example 2. Write an equation for the line in $\mathbb{R}^2$ through the point $(1, 4)$ with normal $\vec{n} = (-2, 1)$. Sketch a diagram indicating the point, the normal vector, and the line.

The same idea can be used to write equations for planes in $\mathbb{R}^3$.

Example 3. Write an equation for the plane in $\mathbb{R}^3$ through the point $(2, -5, 0)$ with normal $\vec{n} = (1, 3, -1)$.

In Chapter 1, we introduced (orthogonal) projections onto the coordinate axes as examples or linear transformations. We can now extend this idea to (orthogonal) projections onto any line in $\mathbb{R}^n$.

Projection Theorem. If $\vec{u}$ and $\vec{a}$ are vectors in $\mathbb{R}^n$ with $\vec{a} \neq \vec{0}$, then $\vec{u}$ can be written in exactly one way as $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\vec{w}_1$ is parallel to $\vec{a}$ and $\vec{w}_2$ is orthogonal to $\vec{a}$. Specifically:

$$\vec{w}_1 = \text{proj}_{\vec{a}}\vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\,\vec{a} \qquad \text{and} \qquad \vec{w}_2 = \vec{u} - \text{proj}_{\vec{a}}\vec{u} = \vec{u} - \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\,\vec{a}.$$

Example 4. Let $\vec{u} = (1, 2, 3)$ and $\vec{a} = (4, -1, -1)$. Find the component of $\vec{u}$ parallel to $\vec{a}$ and the component of $\vec{u}$ orthogonal to $\vec{a}$.

The norm of the orthogonal projection (of $\vec{u}$ onto $\vec{a}$ can be written either in terms of the two vectors or in terms of $\vec{u}$ and the angle θ between $\vec{u}$ and $\vec{a}$.

Example 5. Let L be a line through the origin in $\mathbb{R}^2$ that makes an angle θ with the positive x-axis.

(a) Find the projections of $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$ onto L.

(b) Find the standard matrix P_θ for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that projects each point onto L.

We can use the previous example to find a linear transformation that reflects a vector/point about a line through the origin in $\mathbb{R}^2$.

Example 6. Let $\vec{x} = (4, 1)$ and let L be the line through the origin that makes an angle of $\pi/3$ with the positive x-axis.

(a) Find the projection of $\vec{x}$ onto L.

(b) Find the reflection of $\vec{x}$ about L.

Distance problems.

The distance between a point and a line in $\mathbb{R}^2$ or between a point and a plane in $\mathbb{R}^3$ can be found using projections.

Theorem.

1. In $\mathbb{R}^2$, the distance between the point $P_0 = (x_0, y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

2. In $\mathbb{R}^3$, the distance between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Proof of 2.

Example 7. Find the distance in $\mathbb{R}^2$ between the point $(1, -1)$ and the line $x + 2y = 3$.