Section 3.2: Norm, Dot Product, and Distance in $\mathbb{R}^n$

Objectives.

- Define and apply the notions of norm and distance in $\mathbb{R}^n$.
- Introduce the dot product of two vectors, and interpret the dot product geometrically.
- Study some properties and applications of the dot product.

The norm (length, magnitude) of a vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

Dividing a (non-zero) vector $\vec{v}$ by its norm produces the unit vector in the same direction as $\vec{v}$.

Example 1. Find the unit vector $\vec{u}$ that has the same direction as $\vec{v} = (2, 1, -2)$. Check that $\|\vec{u}\| = 1$.

The distance between two points $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

Example 2. Find the distance between the points $\vec{u} = (1, 3, -2, 0, 2)$ and $\vec{v} = (3, 0, 1, 1, -1)$ in $\mathbb{R}^5$.

The dot product of two vectors $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

Example 3. Find the dot product of the vectors $\vec{u} = (1, 3, 2, 4)$ and $\vec{v} = (-1, 1, -2, 1)$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the dot product of two vectors is related to the angle between them. (This can also be generalized to finding "angles" between vectors in higher-dimensional spaces.)

Example 4. Find the angle between the vectors $\vec{u} = (1, 2)$ and $\vec{v} = (3, 1)$.

Example 5. Find the angle between a diagonal and an edge of a cube.

Notice that the dot product of a vector with itself is the square of the norm of the vector.

Properties of the dot product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and k is a scalar, then:

1. $\vec{u} \cdot \vec{v} =$
2. $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} =$
3. $\vec{u} \cdot (\vec{v} + \vec{w}) =$
4. $(\vec{u} + \vec{v}) \cdot \vec{w} =$
5. $k(\vec{u} \cdot \vec{v}) =$
6. $\vec{v} \cdot \vec{v} \geq 0$, and $\vec{v} \cdot \vec{v} = 0$ if and only if $\vec{v} = 0$.

Example 6. Use properties 1 and 3 above to prove property 4.

Example 7. Expand and simplify the vector expression.

$(2\vec{u} + 3\vec{v}) \cdot (3\vec{u} - \vec{v}) =$

There are two important inequalities involving norms and distances in $\mathbb{R}^n$.

Cauchy-Schwarz Inequality. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|.$$

Triangle Inequality. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, then:

(a) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

(b) $d(\vec{u}, \vec{v}) \leq d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v})$

Proof.

Example 8. Suppose that $\|\vec{u}\| = 4$ and $\|\vec{v}\| = 3$. What are the smallest and largest possible values of $\|\vec{u}+\vec{v}\|$?

In plane geometry (that is, in $\mathbb{R}^2$), the sum of the squares of the two diagonals of a parallelogram equals the sum of the squares of the four sides. This result is also true more generally in $\mathbb{R}^n$.

Parallelogram equation for vectors. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).$$

Proof.

Taking the difference of the squares of the two diagonals of a parallelogram instead gives a different expression for the dot product of two vectors.

Theorem. If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2.$$

Proof.