Section 2.3: Properties of Determinants; Cramer's Rule

Objectives.

- Understand how determinants interact with matrix operations.
- Introduce the adjoint of a square matrix.
- Apply Cramer's Rule to solve a linear system.

We have several methods for finding the determinant of a matrix. We now want to find ways to deal with determinants of expressions such as kA, $A + B$, AB, and A^{-1}.

If A is an $n \times n$ matrix, and k is a scalar, then $\det(kA) = k^n \det A$.

Example 1. Confirm the property above for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the scalar k.

If A and B are square matrices of the same size, then $\det(AB) = (\det A)(\det B)$.

Example 2. Confirm the property above for the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$.

If A is an invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

Example 3. Suppose that A is invertible. Use $\det(AB) = (\det A)(\det B)$ to prove that $\det(A^{-1}) = \dfrac{1}{\det A}$.

For most pairs of matrices, the determinant of the sum is **not** the sum of the determinants.

In general, $\det(A+B) \neq \det A + \det B$.

Example 4. Confirm the property above for the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$.

The one situation where the sum of two determinants is useful is when two matrices are almost identical.

Theorem. Let A, B, and C be square matrices that differ only in row i, and suppose that the ith row of C is the sum of the ith row of A and the ith row of B. Then $\det C = \det A + \det B$.

Example 5. Confirm this theorem for the matrices $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

The <u>(classical) adjoint</u> of a square matrix A is formed by transposing the matrix of cofactors.

Example 6. Find the adjoint of $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix}$.

A useful application of the adjoint matrix is finding an inverse.

Theorem. If A is an invertible matrix, then $A^{-1} = \frac{1}{\det A}\,\mathrm{adj}A$.

Example 7. Find the inverse of the matrix A in the previous example.

<u>Cramer's Rule.</u> If A is an $n \times n$ matrix such that $\det A \neq 0$, then the system $A\vec{x} = \vec{b}$ has the unique solution

$$x_1 = \frac{\det A_1}{\det A}, \qquad x_2 = \frac{\det A_2}{\det A}, \qquad \ldots, \qquad x_n = \frac{\det A_n}{\det A},$$

where A_j is obtained by replacing column j of A with the vector $\vec{b}$.

Example 8. Use Cramer's Rule to solve the linear system:

$$\begin{aligned} x_1 \qquad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$