Section 2.2: Evaluating Determinants by Row Reduction Objectives.

- Understand how elementary row operations affect determinants.
- Use row reduction to compute determinants.
- Introduce column operations and apply them to compute determinants.

The "cofactor expansion" method for finding determinants leads to some useful observations.

Theorem. Let A be a square matrix. If A has a row (or column) of zeros, then det A = 0.

Theorem. Let A be a square matrix. Then det $A = \det A^T$.

Theorem. Let A be a square matrix.

(a) If B is obtained by multiplying a row (or column) of A by a scalar k, then $\det B = k \det A$.

(b) If B is obtained by swapping two rows (or columns) of A, then $\det B = -\det A$.

(c) If B is obtained by adding a multiple of one row of A to another (or a multiple of one column of A to another), then det $B = \det A$.

Theorem. Let *E* be an $n \times n$ elementary matrix.

- (a) If E is obtained by multiplying a row of I_n by a scalar k, then det E = k.
- (b) If E is obtained by swapping two rows of I_n , then det E = -1.
- (c) If E is obtained by adding a multiple of one row of I_n to another, then det E = 1.

Theorem. Let A be a square matrix. If two rows (or two columns) of A are proportional, then det A = 0.

Example 1. Find each determinant.

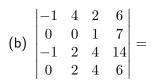
(a) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} =$

(b) $\begin{vmatrix} 1 & 0 & -4 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} =$

	1	$\overline{7}$	3	0	2	
	0	-1	-5	0	0	
(c)	-1	2	-2	0	$-2 \\ 6$	=
	3	0	5	1	6	
	1	$7 \\ -1 \\ 2 \\ 0 \\ 0 \\ 0$	0	0	2	

Example 2. Use row reduction to compute each determinant.

(a)
$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} =$$



We can also use column operations to simplify determinant calculations.

Example 3. Find the determinant of each matrix.

(a)
$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 7 & 0 & -4 \\ 1 & -3 & 3 & 2 \\ 2 & 6 & -5 & 3 \end{bmatrix}$$

(b)
$$B = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Determinants and Solutions of Linear Systems.

In Sections 1.5 and 1.6, we learned about the "Equivalance Theorem", which gives several conditions that are equivalent to a linear system having a unique solution. We can now add a condition involving determinants.

Equivalence Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.

- 1. A is invertible.
- 2. $A\vec{x} = \vec{0}$ has only the trivial solution.
- 3. The reduced row echelon form of A is I_n .
- 4. A can be written as a product of elementary matrices.
- 5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector \vec{b} .
- 6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector \vec{b} .
- 7. det $A \neq 0$

Example 4. Which of the following matrices is invertible?

	[1	0	-2		1	5	1]	$\begin{bmatrix} 0 & 1 \end{bmatrix}$	-1]	[1	0	1]
A =	3	4	1	B =	0	1	6	$C = \begin{vmatrix} -1 & 1 \end{vmatrix}$	-1 D	$\mathcal{P} = 8$	1	-5
A =	0	0	0		0	0	2	0 0	1	$\mathcal{P} = \begin{bmatrix} 1\\ 8\\ 2 \end{bmatrix}$	0	$2 \rfloor$

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \qquad \qquad G = \begin{bmatrix} 1 & 0 & 1 & 5 \\ -4 & 0 & 4 & 1 \\ 0 & 0 & 6 & 2 \\ 2 & 0 & -3 & 1 \end{bmatrix} \qquad \qquad H = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$