Section 2.2: Evaluating Determinants by Row Reduction

Objectives.

- Understand how elementary row operations affect determinants.
- Use row reduction to compute determinants.
- Introduce column operations and apply them to compute determinants.

The "cofactor expansion" method for finding determinants leads to some useful observations.

Theorem. Let A be a square matrix. If A has a row (or column) of zeros, then $\det A = 0$.

Theorem. Let A be a square matrix. Then $\det A = \det A^T$.

Theorem. Let A be a square matrix.

(a) If B is obtained by multiplying a row (or column) of A by a scalar k, then $\det B = k \det A$.

(b) If B is obtained by swapping two rows (or columns) of A, then $\det B = -\det A$.

(c) If B is obtained by adding a multiple of one row of A to another (or a multiple of one column of A to another), then $\det B = \det A$.

Theorem. Let E be an $n \times n$ elementary matrix.

(a) If E is obtained by multiplying a row of I_n by a scalar k, then $\det E = k$.

(b) If E is obtained by swapping two rows of I_n, then $\det E = -1$.

(c) If E is obtained by adding a multiple of one row of I_n to another, then $\det E = 1$.

Theorem. Let A be a square matrix. If two rows (or two columns) of A are proportional, then $\det A = 0$.

Example 1. Find each determinant.

(a) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} =$

(b) $\begin{vmatrix} 1 & 0 & -4 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} =$

(c) $\begin{vmatrix} 1 & 7 & 3 & 0 & 2 \\ 0 & -1 & -5 & 0 & 0 \\ -1 & 2 & -2 & 0 & -2 \\ 3 & 0 & 5 & 1 & 6 \\ 1 & 0 & 0 & 0 & 2 \end{vmatrix} =$

Example 2. Use row reduction to compute each determinant.

(a) $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} =$

(b) $\begin{vmatrix} -1 & 4 & 2 & 6 \\ 0 & 0 & 1 & 7 \\ -1 & 2 & 4 & 14 \\ 0 & 2 & 4 & 6 \end{vmatrix} =$

We can also use column operations to simplify determinant calculations.

Example 3. Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 7 & 0 & -4 \\ 1 & -3 & 3 & 2 \\ 2 & 6 & -5 & 3 \end{bmatrix}$

(b) $B = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$

Determinants and Solutions of Linear Systems.

In Sections 1.5 and 1.6, we learned about the "Equivalance Theorem", which gives several conditions that are equivalent to a linear system having a unique solution. We can now add a condition involving determinants.

Equivalence Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.
5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.
7. $\det A \neq 0$

Example 4. Which of the following matrices is invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & -5 \\ 2 & 0 & 2 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 & 1 & 5 \\ -4 & 0 & 4 & 1 \\ 0 & 0 & 6 & 2 \\ 2 & 0 & -3 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$