Section 2.1: Determinants by Cofactor Expansion

Objectives.

- Understand how to find minors and cofactors.
- Use minors and cofactors to compute the determinant of a square matrix.
- Find the determinant of a 3×3 matrix efficiently.

Recall that the determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

We will use this to *inductively/recursively* define determinants for larger square matrices.

If $A = [a_{ij}]$ is a square matrix, then

- the minor of a_{ij} is
- the cofactor of a_{ij} is

Example 1. Let $A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \\ -1 & 8 & 2 \end{bmatrix}$.

(a) Find the minor of a_{11} and the cofactor of a_{11}.

(b) Find the minor of a_{23} and the cofactor of a_{23}.

Cofactor Expansion.

If A is an $n \times n$ matrix, then the determinant of A is

Example 2. Write out the cofactor expansion of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ along the first column.

Example 3. Find the determinant of the matrix $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{bmatrix}$.

Example 4. Find the determinant of the matrix $C = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 1 & 0 & 5 & 2 \\ -1 & 1 & 0 & 3 \end{bmatrix}$.

Theorem. The determinant of an upper triangular matrix, a lower triangular matrix, or a diagonal matrix is the product of the diagonal entries.

Example 5. Show that the theorem above holds for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$.

Finding determinants can be very time-consuming, especially for large matrices. There is an efficient method for computing the determinant of a 3×3 matrix (without using cofactor expansion) that is similar to how we compute the determinant of a 2×2 matrix.

Example 6. Find the determinant of $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{bmatrix}$.

Example 7. Find all values of λ for which the determinant of $A = \begin{bmatrix} \lambda+1 & 1 \\ 4 & \lambda-2 \end{bmatrix}$ is 0.

So … what is a determinant?

In some sense, the determinant of a square matrix A is a scaling factor for the linear transformation T_A. For instance, if A is a 2×2 matrix, then (the absolute value of) $\det A$ is the area of the parallelogram obtained by applying T_A to the unit square.

Example 8. Consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Find $\det A$, $\det B$, and $\det C$.

(b) Sketch the image of the unit square under the transformations T_A, T_B, and T_C.

(c) Compare the determinants in part (a) with each image in part (b).