Section 3.3: Orthogonal projections in $\mathbb{R}^3$

The orthogonal projections of a vector $\vec{x} = (x, y, z)$ in $\mathbb{R}^3$ onto each of the coordinate axes are given by:

$$
\begin{aligned}
T_x(\vec{x}) &= (x, 0, 0) \qquad \textit{projection onto } x\textit{-axis},\\
T_y(\vec{x}) &= (0, y, 0) \qquad \textit{projection onto } y\textit{-axis},\\
T_z(\vec{x}) &= (0, 0, z) \qquad \textit{projection onto } z\textit{-axis}.
\end{aligned}
$$

Problem 1. Let $\vec{x} = (x, y, z)$ be a vector in $\mathbb{R}^3$.

(a) Show that the vectors $T_x(\vec{x})$ and $T_y(\vec{x})$ are orthogonal.

(b) Show that the vectors $T_x(\vec{x})$ and $\vec{x} - T_x(\vec{x})$ are orthogonal.

(c) Sketch a diagram showing $\vec{x}$, $T_x(\vec{x})$, and $\vec{x} - T_x(\vec{x})$.

Section 3.4: Transformations of lines in $\mathbb{R}^n$

Recall that a line in $\mathbb{R}^n$ can be represented by the equation

$$\vec{x} = \vec{x}_0 + t\vec{v},$$

where $\vec{x}$ is a general point on the line, $\vec{x}_0$ is a fixed point on the line, and $\vec{v}$ is a nonzero vector parallel to the line.

Problem 2. Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator, so that A is an invertible $n \times n$ matrix.

(a) Show that the image of the line $\vec{x} = \vec{x}_0 + t\vec{v}$ in $\mathbb{R}^n$ under the transformation T_A is also a line in $\mathbb{R}^n$.

(b) Let $A = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$. Find vector and parametric equations for the image of the line $\vec{x} = (1, 3) + t(2, -1)$ under multiplication by A.

Section 10.1: Constructing Curves and Surfaces Through Specified Points

Lines in $\mathbb{R}^2$

Any two distinct points (x_1, y_1), (x_2, y_2) in $\mathbb{R}^2$ lie a (unique) line $c_1x + c_2y + c_3 = 0$, where at least one of c_1 and c_2 is not zero. This implies that the homogeneous linear system

$$\begin{aligned} xc_1 + \ \ yc_2 + c_3 &= 0 \\ x_1c_1 + y_1c_2 + c_3 &= 0 \\ x_2c_1 + y_2c_2 + c_3 &= 0 \end{aligned}$$

has a non-trivial solution; equivalently the determinant of the coefficient matrix is zero, which gives the following equation for the line through (x_1, y_1) and (x_2, y_2).

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Problem 3. Consider the line in $\mathbb{R}^2$ through the two points $(3, 1)$ and $(5, -8)$.

(a) Use the determinant above to find an equation for the line.

(b) Find the points where the line intersects each of the coordinate axes.

(c) Graph the equation from part (a) to confirm that the line passes through the two given points.

Circles in $\mathbb{R}^2$

The same method can be used to find a determinant equation for the unique circle

$$c_1(x^2 + y^2) + c_2x + c_3y + c_4 = 0$$

through three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) not on the same line.

Problem 4. Suppose the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) all lie on the circle $c_1(x^2 + y^2) + c_2x + c_3y + c_4 = 0$.

(a) Set up a homogeneous system of linear equations in c_1, c_2, c_3, and c_4 satisfied by the three given points and a general point (x, y) on the same circle.

(b) The system in part (a) has non-trivial solutions. Write a determinant equation to represent this.

(c) Find the center and the radius of the circle passing through $(2, -2)$, $(3, 5)$, and $(4, -6)$.

(d) Graph the equation from part (c) to confirm that the circle passes through the three given points.

Conic sections in $\mathbb{R}^2$

A general conic section in $\mathbb{R}^2$ has equation

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0,$$

and is determined by five distinct points in the plane.

Problem 5. (a) Find a determinant equation for the conic section through the five distinct points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5).$$

(b) Find an equation for the conic section through the points $(0, 0)$, $(0, -1)$, $(2, 0)$, $(2, -5)$, and $(4, -1)$.

(c) Graph the equation from part (b). What type of conic section is this?

Planes in $\mathbb{R}^3$

A plane in $\mathbb{R}^3$ has the scalar equation $c_1x + c_2y + c_3z + c_4 = 0$, and is determined by three points not on the same line.

Problem 6. (a) Find a determinant equation for the plane through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

(b) Find a scalar equation of the plane through the points $(2, 1, 3)$, $(2, -1, -1)$, and $(1, 1, 2)$.

(c) Graph the equation from part (c) to confirm that the plane passes through the three given points.

Spheres in $\mathbb{R}^3$

A sphere in $\mathbb{R}^3$ has equation

$$c_1(x^2 + y^2 + z^2) + c_2x + c_3y + c_4z + c_5 = 0,$$

and is determined by four points not in the same plane.

Problem 7. (a) Find a determinant equation for the sphere through the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4).

(b) Find an equation of the sphere through the points $(0, 1, -2)$, $(1, 3, 1)$, $(2, -1, 0)$, and $(3, 1, -1)$.

(c) Graph the equation from part (c) to confirm that the sphere passes through the four given points.